A Universal Magnification Theorem for Higher-Order Caustic Singularities

A. B. Aazami*
Department of Mathematics, Duke University, Science Drive, Durham, NC 27708

A. O. Petters†
Departments of Mathematics and Physics, and Fuqua School of Business, Duke University, Science Drive, Durham, NC 27708

We prove that, independent of the choice of a lens model, the total signed magnification always sums to zero for a source anywhere in the four-image region close to swallowtail, elliptic umbilic, and hyperbolic umbilic caustics. This is a more global and higher-order analog of the well-known fold and cusp magnification relations, in which the total signed magnification in the two-image region of the fold, and the three-image region of the cusp, are both always zero. As an application, we construct a lensing observable for the hyperbolic umbilic magnification relation and compare it with the corresponding observables for the cusp and fold relations using a singular isothermal ellipsoid lens. We demonstrate the greater generality of the hyperbolic umbilic magnification relation by showing how it applies to the fold image doublets and cusp image triplets, and extends to image configurations that are neither. We show that the results are applicable to the study of substructure on galactic scales using observed quadruple images of lensed quasars. The magnification relations are also proved for generic 1-parameter families of mappings between planes, extending their potential range of applicability beyond lensing.

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I. INTRODUCTION

One of the key signatures of gravitational lensing is the occurrence of multiple images of lensed sources. The magnifications of the images in turn are also known to obey certain relations. One of the simplest examples of a magnification relation is that due to a single point-mass lens, where the two images of the source have signed magnifications that sum to unity: $\mu_1 + \mu_2 = 1$ (e.g., Petters et al. 2001 [1, p. 191]). Witt & Mao 1995 [2] generalized this result to a two point-mass lens. They showed that when the source lies inside the caustic curve, a region which gives rise to five lensed images, the sum of the signed magnifications of these images is also unity: $\sum_i \mu_i = 1$, where μ_i is the signed magnification of image i. This result holds independently of the lens's configuration (in this case, the mass of the point-masses and their positions); it is also true for any source position, so long as the source lies inside the caustic (the region that gives rise to the largest number of images). Further examples of magnification relations, involving other families of lens models (N point-masses, elliptical power-law galaxies, etc.), subsequently followed in Rhie 1997 [3], Dalal 1998 [4], Witt & Mao 2000 [5], Dalal & Rabin 2001 [6], and Hunter & Evans 2001 [7]. More recently, Werner 2008 [8] has shown that the relations for the aforementioned family of lens models are in fact topological invariants.

Although the above relations are "global" in that they involve all the images of a given source, they are not *universal* because the relations depend on the specific class of lens model used. However, it is well-known that for a source near a fold or cusp caustic, the resulting images close to the critical curve are close doublets and triplets whose signed magnifications always sum to zero (e.g, Blandford & Narayan 1986 [15], Schneider & Weiss 1992 [10], Zakharov 1999 [16], [1, Chap. 9]):

$$\begin{aligned} \mu_1 + \mu_2 &= 0 \text{ (fold)} , \\ \mu_1 + \mu_2 + \mu_3 &= 0 \text{ (cusp)} . \end{aligned}$$

These magnification relations are "local" and universal. Their locality means that they apply to a subset of the total number of images produced, namely, a close doublet for the fold and close triplet for the cusp, which requires the source to be near the fold and cusp caustics, respectively. Their universality follows from the fact that the relations

*Electronic address: aazami@math.duke.edu
†Electronic address: petters@math.duke.edu

hold for a generic family of lens models. The higher-order caustics beyond folds and cusps that we consider are five generic caustic surfaces or big caustics occurring in a three-parameter space. Slices of the big caustics give rise to five generic caustic metamorphoses (e.g., [1], Chapters 7 and 9). All five caustic metamorphoses occur in gravitational lensing (e.g., Blandford 1990 [9], Petters 1993 [11], Schneider, Ehlers & Falco 1992 [12], and [1]). In addition, the magnification relations for folds and cusps have been shown to provide powerful diagnostic tools for detecting dark substructure on galactic scales using quadruple lensed images of quasars (e.g., Mao & Schneider 1998 [23]; Keeton, Gaudi & Petters 2003 and 2005 [13, 14]).

The aim of this paper is to show that invariants of the following form also hold universally for lensing maps and general mappings with higher-order caustic singularities:

$$\sum_i \mu_i = 0 \ .$$

In particular, we show that such invariants occur not only for folds and cusps, but also for lensing maps with *elliptic umbilic* and *hyperbolic umbilic* caustics, and for general mappings with *swallowtail*, *elliptic umbilic*, and *hyperbolic umbilic* caustics. Specifically, we prove that the total signed magnification of a source at any point in the four-image region of these higher-order caustic singularities, satisfies:

$$\mu_1 + \mu_2 + \mu_3 + \mu_4 = 0 \ .$$

As an application, we use the hyperbolic umbilic to show how such magnification relations can be used for substructure studies of four-image lens galaxies.

The outline of the paper is as follows. Section II reviews the necessary lensing and singular-theoretic terminologies and results. Section III states our main theorem, which is for generic lensing maps and general mappings. In Section IV, the magnification relations are shown to be relevant to the study of dark substructure in galaxies. We also employ a singular isothermal ellipsoid lens to compare the hyperbolic umbilic relations to the fold and cusp ones. The proof of the main theorem is quite long and so is placed in Appendices A and B.

II. BASIC CONCEPTS

A. Lensing Theory

We begin by reviewing the necessary lensing and singular-theory terminologies. The spacetime geometry for gravitational lensing is treated as a perturbation of a Friedmann universe by a "weak field" spacetime. To that end, we regard a gravitational lens as being localized in a very small portion of the sky. Furthermore, we assume that gravity is "weak", so that near the lens it can be described by a Newtonian potential. We also suppose that the lens is static. Respecting these assumptions, the spacetime metric is given by

$$\boldsymbol{g}_{GL} = -\left(1 + \frac{2\phi}{c^2}\right) c^2 d\tau^2 + a(\tau)^2 \left(1 - \frac{2\phi}{c^2}\right) \left(\frac{dR^2}{1-kR^2} + R^2\left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)\right) \ ,$$

where τ is cosmic time, ϕ the time-independent Newtonian potential of the perturbation caused by the lens, k is the curvature constant, and (R, θ, φ) are the coordinates in space. Here terms of order greater than $1/c^2$ are ignored in any calculation involving ϕ.

The above metric is used to derive the time delay function $T_{\mathbf{y}} : L \longrightarrow \mathbb{R}$, which for a single lens plane is given by

$$T_{\mathbf{y}}(\mathbf{x}) = \frac{1}{2}|\mathbf{x} - \mathbf{y}|^2 - \psi(\mathbf{x}) \ ,$$

where $\mathbf{y} = (s_1, s_2) \in S$ is the position of the source on the light source plane $S = \mathbb{R}^2$, $\mathbf{x} = (u, v) \in L$ is the impact position of a light ray on the lens plane $L \subseteq \mathbb{R}^2$, and $\psi : L \longrightarrow \mathbb{R}$ is the gravitational lens potential. As its name suggests, the time delay function gives the time delay of a lensed light ray emitted from a source in S, relative to the arrival time of a light ray emitted from the same source in the absence of lensing. Fermat's principle yields that light rays emitted from a source that reach an observer are realized as critical points of the time delay function. In other words, a *lensed image* of a light source at $\mathbf{y}$ is a solution $\mathbf{x} \in L$ of the equation $(\operatorname{grad} T_{\mathbf{y}})(\mathbf{x}) = \mathbf{0}$, where the gradient is taken with respect to $\mathbf{x}$. When there is no confusion with the mathematical image of a point, we shall follow common practice and sometimes call a lensed image simply an *image*.

The time delay function also induces a *lensing map* $\boldsymbol{\eta} : L \longrightarrow S$, which is defined by

$$\mathbf{x} \longmapsto \boldsymbol{\eta}(\mathbf{x}) = \mathbf{x} - (\operatorname{grad} \psi)(\mathbf{x}) \ .$$

We call $\boldsymbol{\eta}(\mathbf{x}) = \mathbf{y}$ the *lens equation.* Note that $\mathbf{x} \in L$ is a solution of the lens equation if and only if it is a lensed image because $(\mathrm{grad}\, T_{\mathbf{y}})(\mathbf{x}) = \boldsymbol{\eta}(\mathbf{x}) - \mathbf{y}$. Critical points of the lensing map $\boldsymbol{\eta}$ are those $\mathbf{x} \in L$ for which $\det(\mathrm{Jac}\,\boldsymbol{\eta})(\mathbf{x}) = 0$. Generically, the locus of critical points of the lensing map form curves called *critical curves.* The value $\boldsymbol{\eta}(\mathbf{x})$ of a critical point $\mathbf{x}$ under $\boldsymbol{\eta}$ is called a *caustic point.* These typically form curves, but could be isolated points. Examples of caustics are shown in the third column of Figure 1. For a generic lensing scenario, the number of lensed images of a given source can change (by ± 2 for generic crossings) if and only if the source crosses a caustic. The *signed magnification* of a lensed image $\mathbf{x} \in L$ of a light source at $\mathbf{y} = \boldsymbol{\eta}(\mathbf{x}) \in S$ is given by

$$\mu(\mathbf{x}) = \frac{1}{\det(\mathrm{Jac}\,\boldsymbol{\eta})(\mathbf{x})} \ , \tag{1}$$

where we used the fact that $\det(\mathrm{Jac}\,\boldsymbol{\eta}) = \det(\mathrm{Hess}\, \mathrm{T}_{\mathbf{y}})$ for single plane lensing. Considering the graph of the time delay function, its principal curvatures coincide with the eigenvalues of $\mathrm{Hess}\, T_{\mathbf{y}}(\mathbf{x})$. In addition, its Gaussian curvature at $(\mathbf{x}, T_{\mathbf{y}}(\mathbf{x}))$ equals $\det(\mathrm{Hess}\, T_{\mathbf{y}})(\mathbf{x})$. In other words, the magnification of an image $\mathbf{x}$ can be expressed as

$$\mu(\mathbf{x}) = \frac{1}{\mathrm{Gauss}(\mathbf{x}, T_{\mathbf{y}}(\mathbf{x}))} \ , \tag{2}$$

where $\mathbf{y} = \boldsymbol{\eta}(\mathbf{x})$ and $\mathrm{Gauss}(\mathbf{x}, T_{\mathbf{y}}(\mathbf{x}))$ is the Gaussian curvature of the graph of $T_{\mathbf{y}}$ at the point $(\mathbf{x}, T_{\mathbf{y}}(\mathbf{x}))$. Therefore, the magnification relations are also geometric invariants involving the Gaussian curvature of the graph of $T_{\mathbf{y}}$ at its critical points. Readers are referred to [1, Chap. 6] for a full treatment of these aspects of lensing.

B. Higher-Order Caustic Singularities

This section briefly reviews those aspects of the theory of singularities that will be needed for our main theorem. The central theorem we shall employ is actually summarized in Table I below. It is also worth noting that the terms "universal" and "generic" will be used often. Formally, a property is called *generic* or *universal* if it holds for an open, dense subset of mappings in the given space of mappings. Elements of the open, dense subset are then referred to as being *generic* (or *universal*). See [1, Chap. 8] for a discussion of genericity.

We saw in the previous section that the time delay function $T_{\mathbf{y}}(\mathbf{x})$, which can be viewed as a two-parameter family of functions with parameter $\mathbf{y}$, gives rise to the lensing map $\boldsymbol{\eta} : L \longrightarrow \mathbb{R}^2$. The set of critical points of $\boldsymbol{\eta}$ consists of all $\mathbf{x} \in L$ such that $\det(\mathrm{Jac}\,\boldsymbol{\eta})(\mathbf{x}) = 0$. In this two-dimensional setting, a generic lensing map will have only two types of generic critical points: folds and cusps (see [1, Chap. 8]). The fold critical points map over to caustic arcs that abut isolated cusp caustic points; e.g, see the astroid caustic in Figure 1.

Now, let $T_{c,\mathbf{y}}(\mathbf{x})$ denote a family of time delay functions parametrized by the source position $\mathbf{y}$ and $c \in \mathbb{R}$. In the context of gravitational lensing, the parameter c may denote external shear, core radius, redshift, or some other physical input. The three-parameter family $T_{c,\mathbf{y}}(\mathbf{x})$ gives rise to a one parameter family of lensing maps $\boldsymbol{\eta}_c$. Varying c causes the caustic curves in the light source plane S to evolve with c. This traces out a caustic surface, called a *big caustic*, in the three-dimensional space $\mathbb{R} \times \mathbb{R}^2 = \{(c, \mathbf{y})\}$; see Figure 1. Beyond folds and cusps, these surfaces form higher-order caustics that are classified into three *universal* or *generic* types for locally stable families $\boldsymbol{\eta}_c$, namely, *swallowtails, elliptic umbilics*, and *hyperbolic umbilics* (e.g., Arnold 1986 [17] and [1, Chap. 9]). Generic c-slices of these big caustics also yield *caustic metamorphoses;* see Figure 1. Note that the point $\circ$ is a degenerate point of the lensing map $\boldsymbol{\eta}_c$ on the slice $c = 0$.

For the three-parameter family $T_{c,\mathbf{y}}(\mathbf{x})$ of time delay functions, the universal *quantitative* form of the lensing map can be derived locally using *rigid* coordinate transformations and Taylor expansions, along appropriate constraint equations for the caustics (see [12, Chap. 6] for details). Table I summarizes the quantitative forms of $\boldsymbol{\eta}_c$ for the elliptic umbilic and hyperbolic umbilic critical points. The quantitative form for the swallowtail will be dealt with in future work. Observe that the elliptic and hyperbolic umbilics for $T_{c,\mathbf{y}}$ (or $\boldsymbol{\eta}_c$) do not depend on the lens potential, apart perhaps from c in the event that c is a lens parameter.

One can also consider a general, smooth three-parameter family $F_{c,\mathbf{s}}(\mathbf{x})$ of functions on an open subset of $\mathbb{R}^2$ that induces a one-parameter family of mappings $\mathbf{f}_c$ between planes, which are analogs of the lensing map. The *universal* form (also known as the *generic* or *qualitative* form) of the one-parameter family $\mathbf{f}_c$ is obtained basically by using differentiable equivalence classes of $F_{c,\mathbf{s}}$ that distinguish c from the coordinates of $\mathbf{s}$, to construct catastrophe manifolds that are projected into the space $\{(c, \mathbf{s})\} = \mathbb{R} \times \mathbb{R}^2$ to obtain local coordinate expressions for $\mathbf{f}_c$ (e.g., Majthay 1985 [18], Castrigiano & Hayes 1993 [19], Golubitsky & Guillemin 1973 [20]). These projections of the catastrophe manifolds are called *catastrophe maps* or *Lagrangian maps*, and they are differentiably equivalent to $\mathbf{f}_c$ (see [1, pp. 273-275]). Similar to the case for $T_{c,\mathbf{y}}$ and its induced lensing map $\boldsymbol{\eta}_c$, a generic family $F_{c,\mathbf{s}}$ and its associated map $\mathbf{f}_c$ has three types of caustic singularities beyond folds and cusps: *swallowtails, elliptic umbilics*, and *hyperbolic umbilics* (e.g., [17], [1,

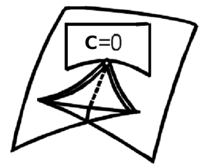
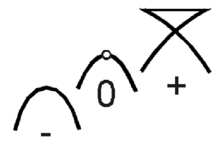
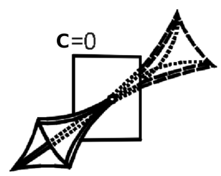
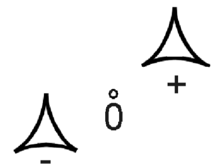
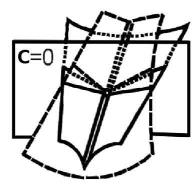
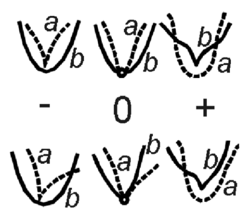
Type	Big Caustic	Caustic Metamorphosis
Swallowtail		
Elliptic Umbilic		
Hyperbolic Umbilic		

FIG. 1: The swallowtail, elliptic umbilic, and hyperbolic umbilic are higher-order caustics shown as surfaces or big caustics in the three-parameter space $\{(c, \mathbf{y})\}$ (middle column). Each c-slice of a big caustic yields caustic curves, which for generic slices evolve according to the metamorphoses in the rightmost column. The point $\circ$ occurs for the slice $c = 0$. In the case of the hyperbolic umbilic, note that the a and b caustic curves are exchanged when c varies through $c = 0$. This classification is due to Arnold 1986 [17].

Chap. 9]). The generic forms of $\mathbf{f}_c$ about fold, cusp, elliptic umbilic, hyperbolic umbilic, and swallowtail singularities are shown in Table I. A detailed treatment of these issues is given in [1, 12, 18].

In summary, the central result about caustic singularities that we shall use can be stated as follows:

- A *generic, smooth* three-parameter family of time delay functions $T_{c,\mathbf{y}}$ can be transformed in a neighborhood of a caustic into one of the forms in the second column of Table I using *rigid* coordinate transformations that distinguish c from the component parameters of $\mathbf{y}$ [1, 12].
- A *generic, smooth* three-parameter family of general functions $F_{c,\mathbf{s}}(\mathbf{x})$, which need not be a time delay family, can be transformed in a neighborhood of a caustic into one of the forms in the third column of Table I using coordinate transformations distinguishing c from the parameters of $\mathbf{s}$ [1, 18].

Caustic	Quantitative Lensing Map	Generic Map
Fold (2D)	$T_{\mathbf{y}}(u,v) = \frac{1}{2}\mathbf{y}^2 - \mathbf{x}\cdot\mathbf{y} + \frac{1}{2}a_{11}u^2 + \frac{1}{6}a_{111}u^3$ $+\frac{1}{2}a_{112}u^2v + \frac{1}{2}a_{122}uv^2 + \frac{1}{6}a_{222}v^3$ $\boldsymbol{\eta}(u,v) = \big(a_{11}u + \frac{1}{2}a_{122}v^2 + a_{112}uv\ ,$ $\frac{1}{2}a_{112}u^2 + a_{122}uv + \frac{1}{2}a_{222}v^2\big)$	$F_{\mathbf{s}}(u,v) = s_1u + s_2v - \frac{1}{2}u^2 - \frac{1}{3}v^3$ $\mathbf{f}(u,v) = \big(u\ ,\ v^2\big)$
Cusp (2D)	$T_{\mathbf{y}}(u,v) = \frac{1}{2}\mathbf{y}^2 - \mathbf{x}\cdot\mathbf{y} + \frac{1}{2}a_{11}u^2 + \frac{1}{6}a_{111}u^3$ $+\frac{1}{2}a_{112}u^2v + \frac{1}{2}a_{122}uv^2 + \frac{1}{24}a_{2222}v^4$ $\boldsymbol{\eta}(u,v) = \big(a_{11}u + \frac{1}{2}a_{122}v^2\ ,\ a_{122}uv + \frac{1}{6}a_{2222}v^3\big)$	$F_{\mathbf{s}}(u,v) = s_1u + s_2v - \frac{1}{2}u^2 - \frac{1}{2}s_1v^2 - \frac{1}{4}v^4$ $\mathbf{f}(u,v) = \big(u\ ,\ uv + v^3\big)$
Elliptic Umbilic (3D)	$T_{c,\mathbf{y}}(u,v) = \frac{1}{2}\mathbf{y}^2 - \mathbf{x}\cdot\mathbf{y} + \frac{1}{3}u^3 - uv^2 + 2cv^2$ $\boldsymbol{\eta}_c(u,v) = \big(u^2 - v^2\ ,\ -2uv + 4cv\big)$	$F_{c,\mathbf{s}}(u,v) = s_1u + s_2v + c(u^2+v^2)$ $+u^3 - 3uv^2$ $\mathbf{f}_c(u,v) = \big(3v^2 - 3u^2 - 2cu\ ,\ 6uv - 2cv\big)$
Hyperbolic Umbilic (3D)	$T_{c,\mathbf{y}}(u,v) = \frac{1}{2}\mathbf{y}^2 - \mathbf{x}\cdot\mathbf{y} + \frac{1}{3}(u^3+v^3) + 2cuv$ $\boldsymbol{\eta}_c(u,v) = \big(u^2 + 2cv\ ,\ v^2 + 2cu\big)$	$F_{c,\mathbf{s}}(u,v) = s_1u + s_2v + cuv + u^3 + v^3$ $\mathbf{f}_c(u,v) = \big(-3u^2 - cv\ ,\ -3v^2 - cu\big)$
Swallowtail (3D)		$F_{c,\mathbf{s}}(u,v) = s_1u + s_2v - \frac{1}{2}s_2u^2 - \frac{1}{2}v^2$ $-\frac{1}{3}cu^3 - \frac{1}{5}u^5$ $\mathbf{f}_c(u,v) = \big(uv + cu^2 + u^4\ ,\ v\big)$

TABLE I: For each type of caustic singularity, the second and third columns show the respective universal local forms of the smooth three-parameter family of time delay functions $T_{c,\mathbf{y}}$ and family of general functions $F_{c,\mathbf{s}}$, along with their one-parameter family of lensing maps $\boldsymbol{\eta}_c$ and induced general maps $\mathbf{f}_c$. For the two-parameter case of the fold and cusp, the constants a_{ijk} denote partial derivatives of $T_{c,\mathbf{y}}(\mathbf{x})$ with respect to $\mathbf{x} = (u,v) \equiv (x_1, x_2)$, evaluated at the origin: $a_{ijk} = (\partial^3 T_{c,\mathbf{y}}/\partial x_i \partial x_j \partial x_k)(\mathbf{0})$. These constants do not appear in the quantitative forms of the elliptic and hyperbolic umbilic in the second column. This implies that the local behavior of $T_{c,\mathbf{y}}$ and $\boldsymbol{\eta}_c$ about an elliptic or hyperbolic umbilic does not depend on the lens potential, except possibly through c when c is a lens parameter. We have omitted the quantitative form of the swallowtail for $T_{c,\mathbf{y}}$ and $\boldsymbol{\eta}_c$ because the proof of its magnification relation will appear in forthcoming work.

III. MAIN THEOREM

Consider the universal one-parameter family of lensing maps $\boldsymbol{\eta}_c$ in Table I. Let $\mathbf{x}_i$ denote a lensed image of a source at $\mathbf{y}$, that is, $\mathbf{y} = \boldsymbol{\eta}_c(\mathbf{x}_i)$, and let μ_i be the magnification of $\mathbf{x}_i$, which by (1) is $\mu_i = 1/\det(\mathrm{Jac}\ \boldsymbol{\eta}_c)(\mathbf{x}_i)$. For the generic mappings $\mathbf{f}_c$ in Table I, we define the analog of magnification as follows:

$$\mathfrak{M}_i = \frac{1}{\det(\mathrm{Jac}\,\mathbf{f}_c)(\mathbf{x}_i)}\ ,$$

where $\mathbf{f}_c(\mathbf{x}_i) = \mathbf{s}$ or, equivalently, the point $(\mathbf{x}_i, F_{c,\mathbf{s}}(\mathbf{x}_i))$ is a critical point in the graph of $F_{c,\mathbf{s}}$.

Theorem 1. *For any of the smooth generic three-parameter family of time delay functions $T_{c,\mathbf{y}}$ (or lensing maps $\boldsymbol{\eta}_c$)and family of general functions $F_{c,\mathbf{s}}$ (or general mappings $\mathbf{f}_c$) in Table I, and for any source position $\mathbf{y}$ and point $\mathbf{s}$ in the indicated region, the following results hold:*

1. A_2 (Fold) *Magnification relations in two-image region:*

$$\mu_1 + \mu_2 = 0 \ , \qquad \mathfrak{M}_1 + \mathfrak{M}_2 = 0 \ .$$

2. A_3 (Cusp) *Magnification relations in three-image region:*

$$\mu_1 + \mu_2 + \mu_3 = 0 \ , \qquad \mathfrak{M}_1 + \mathfrak{M}_2 + \mathfrak{M}_3 = 0 \ .$$

3. A_4 (Swallowtail) *Magnification relation in four-image region:*

$$\mathfrak{M}_1 + \mathfrak{M}_2 + \mathfrak{M}_3 + \mathfrak{M}_4 = 0 \ .$$

4. D_4^- (Elliptic Umbilic) *Magnification relations in four-image region:*

$$\mu_1 + \mu_2 + \mu_3 + \mu_4 = 0 \ , \qquad \mathfrak{M}_1 + \mathfrak{M}_2 + \mathfrak{M}_3 + \mathfrak{M}_4 = 0 \ .$$

5. D_4^+ (Hyperbolic Umbilic) *Magnification relations in four-image region:*

$$\mu_1 + \mu_2 + \mu_3 + \mu_4 = 0 \ , \qquad \mathfrak{M}_1 + \mathfrak{M}_2 + \mathfrak{M}_3 + \mathfrak{M}_4 = 0 \ .$$

In the theorem, the μ-magnification (resp., $\mathfrak{M}$-magnification) relations are *universal* or *generic* in the sense that they hold for an open, dense set of three-parameter families $T_{c,\mathbf{y}}$ (resp., general families $F_{c,\mathbf{s}}$) in the space of such families; see [17] and [1, Chaps. 7,8]. Readers are referred to [1, Chap. 8] for a discussion of universality/genericity.

The magnification relations in Theorem 1 are also geometric invariants. In fact, we saw in equation (2) that each μ_i is a reciprocal of the Gaussian curvature. This is also true of the quantities $\mathfrak{M}_i$. To see this, recall that the Gaussian curvature at the point $(\mathbf{x}_i, F_{c,\mathbf{s}}(\mathbf{x}_i))$ in the graph of $F_{c,\mathbf{s}}$ is given by

$$\mathrm{Gauss}(\mathbf{x}_i, F_{c,\mathbf{s}}(\mathbf{x}_i)) = \frac{\det(\mathrm{Hess}\, F_{c,\mathbf{s}})(\mathbf{x}_i)}{1 + |\,\mathrm{grad}\, F_{c,\mathbf{s}}(\mathbf{x}_i)|^2} \ .$$

But $(\mathbf{x}_i, F_{c,\mathbf{s}}(\mathbf{x}_i))$ is a critical point of the graph, so $\mathrm{grad}\, F_{c,\mathbf{s}}(\mathbf{x}_i) = \mathbf{0}$. A computation also shows that

$$\det(\mathrm{Jac}\, \mathbf{f}_c) = \det(\mathrm{Hess}\, F_{c,\mathbf{s}}) \ .$$

Hence

$$\mathfrak{M}_i = \frac{1}{\mathrm{Gauss}(\mathbf{x}_i, F_{c,\mathbf{s}}(\mathbf{x}_i))} \ .$$

We use the A, D classification notation of Arnold 1973 [21] in the theorem. This notation highlights a deep link between the above singularities and Coxeter-Dynkin diagrams appearing in the theory of simple Lie algebras. Theorem 1 is also apparently related to a deep result in singularity theory, namely, the inverse Jacobian Theorem and its corollary, the Euler-Jacobi formula (see Arnold, Gusein-Zade, & Varchenko 1985 [22]). We are thankful to the referee for pointing out this link, which is currently being pursued by the authors.

As mentioned in the introduction, the fold and cusp magnification relations are known [1, 10, 15, 16], but we restate them in the theorem for completeness. In addition, note that the magnification relation for the swallowtail is established only for the generic form; the quantitative lensing case will be taken up in future work.

The proof of Theorem 1 is very long. Appendix A gives a detailed proof of the μ-magnification relations, while Appendix B provides a proof of the $\mathfrak{M}$-magnification relations.

IV. APPLICATIONS

Before discussing the applications, we recall that the magnification μ_i of a lensed image is the flux F_i of the image divided by the flux F_S of the unlensed source (e.g., [1, pp. 82-85]):

$$\mu_i = \pm \frac{F_i}{F_S} \ ,$$

where the "+" choice is for even index images (minima and maxima) and the "−" choice is for odd index images (saddles). Though F_i is an observable, the source's flux F_S is generally unknown. Consequently, the magnification μ_i is not directly observable and so magnification sums $\sum_i \mu_i$ are also not observable. However, we can construct an observable by introducing the following quantity:

$$R \equiv \frac{\sum_i \mu_i}{\sum_i |\mu_i|} = \frac{\sum_i (\pm) F_i}{\sum_i F_i} \ , \tag{3}$$

where the $\pm$ choice is the same as above. This quantity is in terms of the observable image fluxes F_i and image signs, which can be determined for real systems [13, 14].

Now, aside from their natural theoretical interest, the importance of magnification relations in gravitational lensing arises in their applications to detecting dark substructure in galaxies using "anomalous" flux ratios of multiply imaged quasars. The setting consists typically of four images of a quasar lensed by a foreground galaxy. The *smooth* mass density models used for the galaxy lens usually accurately reproduce the number and relative positions of the images, but fail to reproduce the image flux ratios. For the case of a cusp, where a close image triplet appears, Mao & Schneider 1998 [23] showed that the cusp μ-magnification relation fails (i.e., deviates from zero) and argued that it does so since the *smoothness assumption* about the galaxy lens breaks down on the scale of the fold image doublet. In other words, a violation of the cusp magnification relation in a real lens system implies a violation of smoothness in the lens, which in turn invokes the presence of substructure or graininess in the galaxy lens on the scale of the image separation. Soon thereafter Metcalf & Madau 2001 [24] and Chiba 2002 [25] showed that dark matter was a plausible candidate for this substructure.

In 2003 and 2005, Keeton, Gaudi & Petters [13, 14] developed a rigorous theoretical framework showing how the fold and cusp μ-magnification relations provide a diagnostic for detecting substructure on galactic scales. Their analysis employs the R-quantity (3) for folds and cusps:

$$R_{\rm fold} \equiv \frac{\mu_1 + \mu_2}{|\mu_1| + |\mu_2|} = \frac{F_1 - F_2}{F_1 + F_2} \ , \qquad\qquad R_{\rm cusp} \equiv \frac{\mu_1 + \mu_2 + \mu_3}{|\mu_1| + |\mu_2| + |\mu_3|} = \frac{F_1 - F_2 + F_3}{F_1 + F_2 + F_3} \ ,$$

where F_i is the observable flux of image i and image 2 has negative parity. For a source sufficiently close to a fold (resp., cusp) caustic, the images will have a close image pair (resp., close image triplet); see the close doublets and triplets in Figure 2(a,b,d,e). Theoretically, these images should have vanishing $R_{\rm fold}$ and $R_{\rm cusp}$ due to the fold and cusp magnification relations and so nontrivial deviations from zero would signal the presence of substructure. In [13, 14], it was shown that 5 of the 12 fold-image systems and 3 of the 4 cusp-image ones showed evidence for substructure.

The study above would look at a multiple-image system and consider subsets of two and three images to analyze $R_{\rm fold}$ and $R_{\rm cusp}$, respectively. Such analyses are then "local" when more than three images occur since only two or three images are studied at a time. Theorem 1 generalizes the above R-quantities from folds and cusps to generic smooth lens systems that exhibit swallowtail, elliptic umbilic, and hyperbolic umbilic singularities. The R-quantities resulting from these higher-order singularities allow one to consider *four* images at a time and so are more global than the fold and cusp relations in terms of how many images are incorporated. The singularity that is most applicable to observed quadruple-images produced by the lensing of quasars is the hyperbolic umbilic (cf. Figure 2). The associated R-quantity is

$$R_{\rm h.u.} \equiv \frac{\mu_1 + \mu_2 + \mu_3 + \mu_4}{|\mu_1| + |\mu_2| + |\mu_3| + |\mu_4|} = \frac{F_1 - F_2 + F_3 - F_4}{F_1 + F_2 + F_3 + F_4} \ ,$$

where images 2 and 4 have negative parity.

We now illustrate the hyperbolic umbilic quantity $R_{\rm h.u.}$ using a well-known model for a galaxy lens, namely, a singular isothermal ellipsoid (SIE) lens. The SIE lens potential and surface mass density are given respectively as follows:

$$\psi(r,\varphi) = rF(\varphi) - \frac{\gamma}{2} r^2 \cos 2\varphi, \qquad \kappa(r,\varphi) = \frac{G(\varphi)}{2r},$$

where $F(\varphi)$ and $G(\varphi)$ satisfy $G(\varphi) = F(\varphi) + F''(\varphi)$ by Poisson's equation, and are given explicitly by

$$\begin{aligned} G(\varphi) &= \frac{R_{\rm ein}}{\sqrt{1-\varepsilon\cos 2\varphi}} \ , \\ F(\varphi) &= \frac{R_{\rm ein}}{\sqrt{2\varepsilon}}\left[\cos\varphi\tan^{-1}\left(\frac{\sqrt{2\varepsilon}\cos\varphi}{\sqrt{1-\varepsilon\cos 2\varphi}}\right) + \sin\varphi\tanh^{-1}\left(\frac{\sqrt{2\varepsilon}\sin\varphi}{\sqrt{1-\varepsilon\cos 2\varphi}}\right)\right] , \end{aligned}$$

where $R_{\rm ein}$ is the angular Einstein ring radius. The parameter ε is related to the axis ratio q by $\varepsilon = (1-q^2)/(1+q^2)$, and should not be confused with the ellipticity $e = 1-q$. The cusp at $\varphi = 0$ is given by

$$\mathbf{y}_{\rm cusp} = \left(\frac{2\gamma F(0) + (1+\gamma)F''(0)}{1-\gamma}\, , \, 0\right) \ . \tag{4}$$

Using the *Gravlens* software by Keeton 2001 [26], we now solve the SIE lens equation for sources on the positive horizontal axis in the four-image region of the light source plane, and compute $R_{\rm h.u.}$. Let the SIE have ellipticity $e = 0.35$ and shear $\gamma = 0.05$ oriented along the horizontal axis; both of these values are observationally motivated [13, 14]. Figure 2(a,b,c) shows three important image configurations for the SIE: the fold, when the source lies close to a fold arc and produces a close pair of images about a critical curve; the cusp, when the source lies close to a cusp caustic and produces a close triplet of images about a critical curve; the cross-like configuration of four images, when the source sits nearer to the center of the astroid-shaped inner caustic curve. Figure 2(d,e,f) illustrates how the SIE image configurations are similar to those of the hyperbolic umbilic lensing map $\boldsymbol{\eta}_c$ given in Table I. See Appendix A 2 for more on the hyperbolic umbilic $\boldsymbol{\eta}_c$.

We now look at the behavior of $R_{\rm fold}$, $R_{\rm cusp}$, and $R_{\rm h.u.}$ for an SIE. Table II compares $R_{\rm fold}$ and $R_{\rm h.u.}$ for a source approaching a fold arc diagonally from the center of the astroid-shaped inner caustic; see Figure 2(a). The fold point where the diagonal intersects the fold arc is at

$$\mathbf{y}_{\rm fold} \approx (0.14055 R_{\rm ein},\ 0.14055 R_{\rm ein}) \ .$$

As the source at $\mathbf{y}$ approaches $\mathbf{y}_{\rm fold}$ along the diagonal, the values in Table II show that $R_{\rm fold}$ and $R_{\rm h.u.}$ each approach the ideal value of 0, and that $R_{\rm h.u.}$ approaches $R_{\rm fold}$ from *above.* The reason for this is as follows: From Figure 2(a) we see that there are two pairs of images in a hyperbolic umbilic configuration: the fold image doublet straddling the critical curve, and whose two images we denote by d_1, d_2, and the pair consisting of the outer two images, which we denote by o_1, o_2. The quantity $R_{\rm h.u.}$ then becomes

$$R_{\rm h.u.} = \frac{|\mu_{d_1}| - |\mu_{d_2}| + |\mu_{o_1}| - |\mu_{o_2}|}{|\mu_{d_1}| + |\mu_{d_2}| + |\mu_{o_1}| + |\mu_{o_2}|} \ .$$

As the source approaches $\mathbf{y}_{\rm fold}$ along the diagonal, Table II shows that the quantities $|\mu_{d_1}| - |\mu_{d_2}|$ and $|\mu_{o_1}| - |\mu_{o_2}|$ stay roughly constant, though the individual magnifications vary. In addition, near the fold, we see that $|\mu_{d_1}| + |\mu_{d_2}|$ dominates $|\mu_{o_1}| + |\mu_{o_2}|$, causing the denominator of $R_{\rm h.u.}$ to approach $|\mu_{d_1}| + |\mu_{d_2}|$, which is the denominator of $R_{\rm fold}$. This leads to

$$R_{\rm h.u.} \approx \frac{|\mu_{d_1}| - |\mu_{d_2}|}{|\mu_{d_1}| + |\mu_{d_2}|} + \frac{|\mu_{o_3}| - |\mu_{o_4}|}{|\mu_{d_1}| + |\mu_{d_2}|} \geq \frac{|\mu_{d_1}| - |\mu_{d_2}|}{|\mu_{d_1}| + |\mu_{d_2}|} = R_{\rm fold} \ .$$

The net effect is that $R_{\rm h.u.}$ approaches $R_{\rm fold}$ from above (at least for the path along the diagonal). Furthermore, since the quantity $|\mu_{d_1}| + |\mu_{d_2}|$ diverges, we see that both $R_{\rm h.u.}$ and $R_{\rm fold}$ approach the magnification relation value of 0.

Table III compares $R_{\rm h.u.}$ with $R_{\rm cusp}$ for a source approaching a cusp along the horizontal axis from the center of the astroid-shaped caustic curve; see Figure 2(b,c). For these values of the ellipticity and shear, we see from (4) that the two cusps on the horizontal axis are located at

$$\mathbf{y}^{\pm}_{\rm cusp} \approx (\pm 0.48 R_{\rm ein}, 0) \ . \tag{5}$$

The table shows that as the source approaches $\mathbf{y}^{+}_{\rm cusp}$ along the horizontal axis, the quantity $R_{\rm h.u.}$ approaches $R_{\rm fold}$ from *below.* In other words, $R_{\rm h.u.}$ is smaller than $R_{\rm fold}$. To see why this happens, consider the triplet of sub-images in Figure 2(b), which we denote by t_1, t_2, t_3, and the extra outer image, denote by o. With this notation,

$$R_{\rm h.u.} = \frac{|\mu_{t_1}| - |\mu_{t_2}| + |\mu_{t_3}| - |\mu_{o}|}{|\mu_{t_1}| + |\mu_{t_2}| + |\mu_{t_3}| + |\mu_{o}|} \ .$$

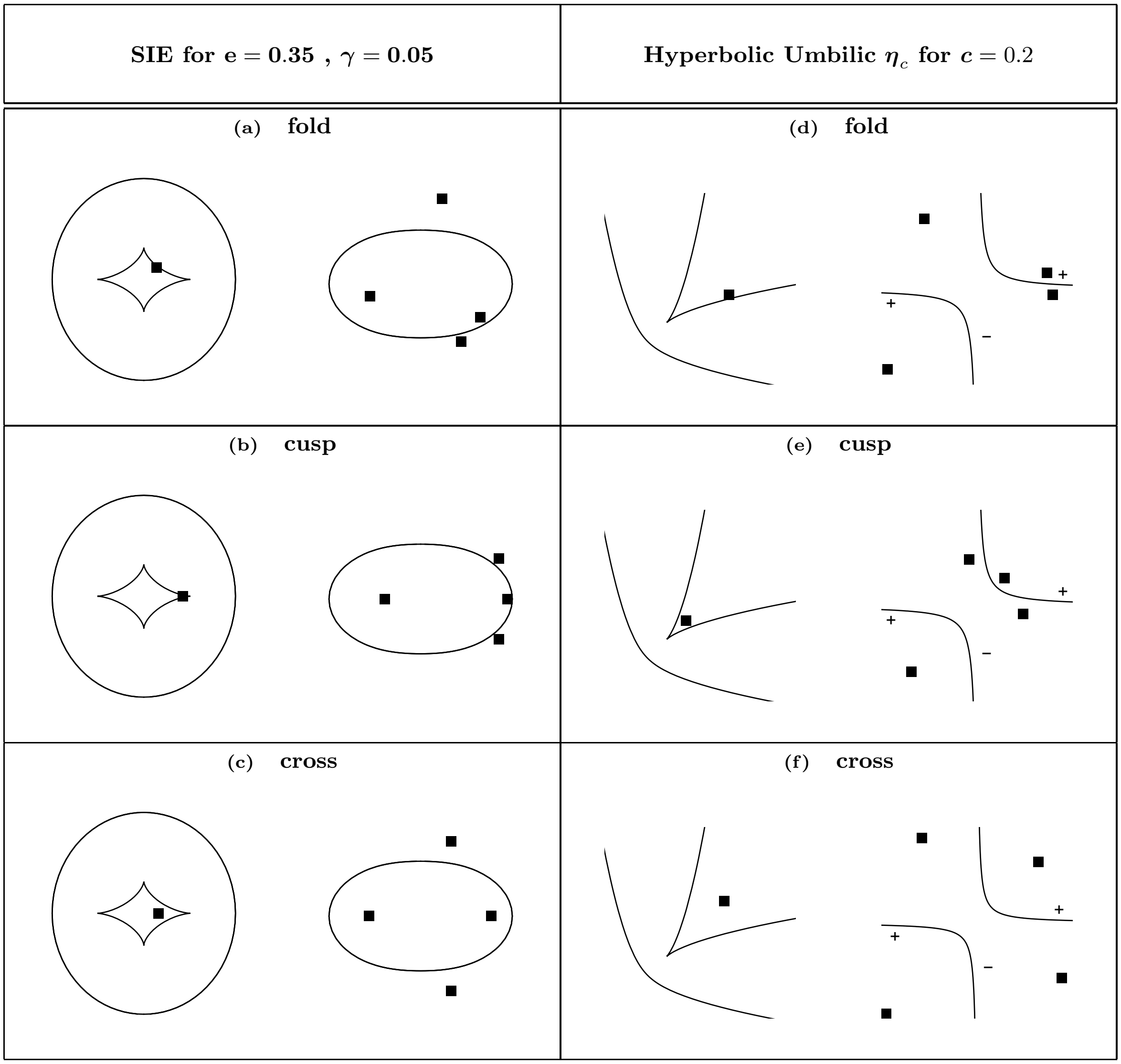


FIG. 2: The first column shows fold, cusp, and cross-like configurations due to an SIE with ellipticity $e = 0.35$ and shear $\gamma = 0.05$ oriented along the horizontal axis (Panels a,b,c). The second column shows the same configurations due to the hyperbolic umbilic lensing map $\boldsymbol{\eta}_c$ in Table I with parameter value $c = 0.2$ (Panels d,e,f). In each panel, the sub-figure on the left depicts the caustic curves with source position (solid box) in the light source plane, while the sub-figure on the right shows the critical curves with image positions (solid boxes) in the lens plane. For the hyperbolic umbilic, image parities have been indicated through $\pm$ in the given regions. Note that the cross-like configuration shown for the SIE is not a perfect cross, which would be the case if the source were centered inside the astroid-shaped inner caustic. Also, for the SIE fold and cusp configurations, the source is actually located inside (rather than over) the cusped curve of the astroid.

As the source approaches $\mathbf{y}^{+}_{\text{cusp}}$ along the horizontal axis, the values in Table III of the cusp relation $|\mu_{t_1}| - |\mu_{t_2}| + |\mu_{t_3}|$ are *positive*. The inclusion of the outer, negative parity magnification μ_o then subtracts from that positive value, yielding

$$(|\mu_{t_1}| - |\mu_{t_2}| + |\mu_{t_3}|) - |\mu_o| \leq |\mu_{t_1}| - |\mu_{t_2}| + |\mu_{t_3}| \ ,$$

which implies that

$$R_{\text{h.u.}} \leq R_{\text{cusp}} \ .$$

Furthermore, Table III shows that $|\mu_o|$ grows fainter faster than the value of the signed magnification of the triplet, which yields

$$|\mu_{t_1}| + |\mu_{t_2}| + |\mu_{t_3}| \ \geq \ |\mu_{t_1}| - |\mu_{t_2}| + |\mu_{t_3}| \gg |\mu_o| \ .$$

Source	R_{fold}	$R_{\text{h.u.}}$	$\lvert\mu_{d_1}\rvert - \lvert\mu_{d_2}\rvert$	$\lvert\mu_{o_1}\rvert - \lvert\mu_{o_2}\rvert$	$\lvert\mu_{d_1}\rvert + \lvert\mu_{d_2}\rvert$	$\lvert\mu_{o_1}\rvert + \lvert\mu_{o_2}\rvert$
$(0.10R_{\text{ein}}\,, 0.10R_{\text{ein}})$	0.14	0.19	1.22	1.21	8.51	4.35
$(0.11R_{\text{ein}}\,, 0.11R_{\text{ein}})$	0.13	0.18	1.22	1.22	9.64	4.28
$(0.12R_{\text{ein}}\,, 0.12R_{\text{ein}})$	0.11	0.15	1.22	1.22	11.55	4.21
$(0.13R_{\text{ein}}\,, 0.13R_{\text{ein}})$	0.08	0.12	1.22	1.22	15.83	4.15
$(0.14R_{\text{ein}}\,, 0.14R_{\text{ein}})$	0.02	0.04	1.21	1.23	65.17	4.081
$(0.1405R_{\text{ein}}\,, 0.1405R_{\text{ein}})$	0.008	0.015	1.21	1.23	156.80	4.078

TABLE II: The quantities $R_{\text{h.u.}}$ and R_{fold} for an SIE with $e = 0.35$ and $\gamma = 0.05$ oriented along the horizontal axis. The source approaches the fold point $\mathbf{y}_{\text{fold}} \approx (0.14055R_{\text{ein}},\ 0.14055R_{\text{ein}})$ diagonally from the center of the astroid-shaped inner caustic. The quantity $|\mu_{d_1}| - |\mu_{d_2}|$ is the difference in the magnifications of the images in the close doublet, while $|\mu_{o_1}| - |\mu_{o_2}|$ is the difference for the remaining two outer images; cf. Figure 2(a).

Source	R_{cusp}	$R_{\text{h.u.}}$	$\lvert\mu_{t_1}\rvert + \lvert\mu_{t_2}\rvert + \lvert\mu_{t_3}\rvert$	$\lvert\mu_{t_1}\rvert - \lvert\mu_{t_2}\rvert + \lvert\mu_{t_3}\rvert$	$\lvert\mu_{o_1}\rvert$
$(0\,, 0)$ (center)	0.52	0.23	8.49	4.46	2.02
$(0.10R_{\text{ein}}\,, 0)$	0.41	0.22	9.58	3.94	1.49
$(0.15R_{\text{ein}}\,, 0)$	0.36	0.21	10.57	3.76	1.29
$(0.20R_{\text{ein}}\,, 0)$	0.30	0.19	12.02	3.61	1.12
$(0.25R_{\text{ein}}\,, 0)$	0.25	0.17	14.20	3.48	0.98
$(0.30R_{\text{ein}}\,, 0)$	0.19	0.14	17.71	3.38	0.85
$(0.35R_{\text{ein}}\,, 0)$	0.14	0.10	24.10	3.30	0.74
$(0.40R_{\text{ein}}\,, 0)$	0.08	0.07	39.02	3.23	0.64
$(0.45R_{\text{ein}}\,, 0)$	0.03	0.02	111.5	3.18	0.55

TABLE III: The quantities $R_{\text{h.u.}}$ and R_{cusp} for an SIE with $e = 0.35$ and $\gamma = 0.05$ oriented along the horizontal axis. The source approaches the cusp point $\mathbf{y}^{+}_{\text{cusp}} \approx (0.48R_{\text{ein}}, 0)$ along the horizontal axis from the center of the astroid-shaped inner caustic. The quantity $|\mu_{t_1}| - |\mu_{t_2}| + |\mu_{t_3}|$ is the signed magnification sum of the cusp triplet, while $|\mu_o|$ is the magnification of the outer image; see Figure 2(b).

In other words, as the source approaches $\mathbf{y}^{+}_{\text{cusp}}$ along the horizontal axis, the contribution of the outer image $|\mu_o|$ to the numerator and denominator of $R_{\text{h.u.}}$ becomes negligible. The net effect, at least for the given horizontal axis approach, is that $R_{\text{h.u.}}$ and R_{cusp} converge, with $R_{\text{h.u.}}$ approaching R_{cusp} from below as they both approach the magnification relation value of 0.

Finally, though $R_{\text{h.u.}}$ can approximate R_{fold} and R_{cusp} for fold image doublets and cusp image triplets, resp., *the hyperbolic umbilic magnification relation has a more global reach* in terms of the number of images included. This is because $R_{\text{h.u.}}$ also applies directly to image configurations that are neither close doublets nor triplets; e.g., to cross-like configurations as in Figure 2(c). For instance, it was determined in [13] that to satisfy the relation $|R_{\text{cusp}}| < 0.1$ at 99% confidence, the opening angle must be $\theta \lesssim 30°$. By opening angle we mean the angle of the polygon spanned by the three images in the cusp triplet, measured from the position of the lens galaxy, which in our case, is centered at the origin in the lens plane. For the SIE cross-like configuration shown in Figure 2(c), the opening angle is $\theta \approx 140°$; a perfect cross, which would be the case if the source were centered inside the astroid-shaped inner caustic, has $\theta = 180°$. In other words, to satisfy the cusp relation reasonably well, the cusp triplet must be quite tight as, for example, in the SIE cusp triplet shown in Figure 2(b). By contrast, the quantity $R_{\text{h.u.}}$ applies even for values $\theta \gg 30°$. (In Table III note how $R_{\text{h.u.}}$ is smaller than R_{cusp} for source positions closer to the center $(0, 0)$, which yield more cross-like image configurations.)

A more detailed study of the properties of $R_{\text{h.u.}}$ would require a separate paper and involve a Monte Carlo analysis similar to that employed in [13, 14] to study R_{cusp} and R_{fold}. Such an analysis of $R_{\text{h.u.}}$ would be applicable to the currently known 26 four-image lens systems (courtesy of the CASTLES lens sample [27])

V. CONCLUSION

We showed that magnification invariants hold universally not only for folds and cusps, but also for swallowtails, elliptic umbilics, and hyperbolic umbilics. Specifically, for a source anywhere in the four-image region close to each

of these caustic singularities, the total signed magnification is identically zero. This result is universal in that it does not depend on the class of lens models used, and is thus an extension of the familiar fold and cusp magnification sum relations. We proved that these relations hold for generic one-parameter families of lensing maps with the elliptic umbilic and hyperbolic umbilic singularities. We also established the relations for generic one-parameter families of general mappings, which need not relate to lensing, for the swallowtail, elliptic umbilic, and hyperbolic umbilic singularities. We emphasized that these universal sum relations are *geometric* invariants, because they are sums of reciprocals of Gaussian curvatures at critical points.

The relevance of these higher order magnification invariants to the study of dark substructure in galaxies was shown. Using a singular isothermal ellipsoidal model of a galaxy lens, we constructed a lensing observable for the hyperbolic umbilic, denoted $R_{\rm h.u.}$, and compared it to the well-known fold and cusp analogues, $R_{\rm fold}$ and $R_{\rm cusp}$. These three observables approach their magnification relation value of 0 the closer a source gets to a caustic. Significant deviations from this value indicate that the lens in question is not smooth, but has some kind of substructure on the scale of the image separations. We showed that $R_{\rm h.u.}$ is a more global quantity than $R_{\rm fold}$ and $R_{\rm cusp}$ because $R_{\rm h.u.}$ considers four lensed images simultaneously, while $R_{\rm fold}$ considers two and $R_{\rm cusp}$ three. At the same time, we showed that, as the source approached a fold arc or cusp point, the quantity $R_{\rm h.u.}$ approaches $R_{\rm fold}$ or $R_{\rm cusp}$, respectively. More stringent conclusions about the properties of $R_{\rm h.u.}$ await a full Monte Carlo analysis, akin to the one employed recently to examine $R_{\rm fold}$ and $R_{\rm cusp}$.

VI. ACKNOWLEDGMENTS

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APPENDIX A: PROOF OF THE MAIN THEOREM FOR LENSING MAPS

1. Elliptic Umbilic

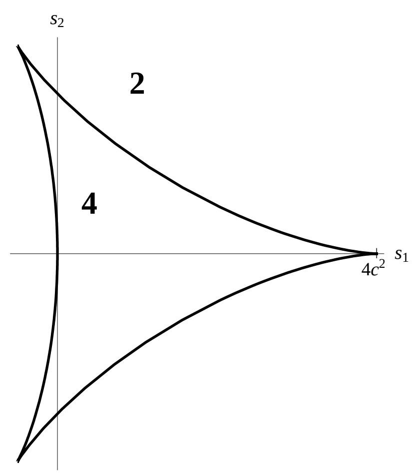

FIG. 3: Caustic curve for an elliptic umbilic lensing map $\boldsymbol{\eta}_c$ in equation (A1) or Table I. The number of lensed images for sources in their respective regions is indicated.

The derivation of the quantitative form of the lensing map in the neighborhood of an elliptic umbilic critical point, can be found in [12, Chap. 6]. The resulting map is

$$\begin{aligned} s_1 &= u^2 - v^2 \ , \\ s_2 &= -2uv + 4cv \ . \end{aligned} \tag{A1}$$

Here $\mathbf{y} = (s_1, s_2)$ is the location of a source on the light source plane S, (u, v) the location of a corresponding lensed image on the lens plane L, and c is a constant which signifies that the lens mapping under consideration is one in a

one-parameter family of lens mappings. The magnification of an image (u, v) is

$$(\det(\text{Jac }\mathbf{s}))^{-1}(u,v) \;=\; \frac{1}{8cu - 4(u^2+v^2)} \;. \tag{A2}$$

A parameter representation of the critical curve is

$$\begin{aligned} u &= c(1+\cos\phi)\;, \\ v &= c\sin\phi\;. \end{aligned}$$

Inserting these into the lens equation (A1) gives the caustic curve:

$$\begin{aligned} s_1 &= 2c^2\cos\phi\,(1+\cos\phi)\;, \\ s_2 &= 2c^2\sin\phi\,(1-\cos\phi)\;. \end{aligned}$$

(Note that our notation differs from that of [12].) The caustic curve is shown in Figure 3. The region inside the closed caustic curve constitutes the four-image region. We now show that for all sources inside this region, the total signed magnification is identically zero.

We begin by considering a special case: sources in the four-image region lying on the horizontal axis; that is, with $s_2 = 0$. In this case the lens equation (A1) is solvable. The lensed images are $\left(\pm\sqrt{s_1}, 0\right), \left(2c, \pm\sqrt{4c^2 - s_1}\right)$, where we note that all four of these images are real because $0 < s_1 < 4c^2$ inside the caustic curve (see Figure 3). The total signed magnification, obtained by inserting each of these four images into (A2) and summing over, will be zero. For the remainder of this section, therefore, we can restrict ourselves to sources (s_1, s_2) inside the caustic curve with $s_2 \neq 0$. Note from the second lens equation (A1) that $s_2 \neq 0$ forces the v-coordinate of each lensed image to be nonzero: $v_i \neq 0$. This fact will prove useful below.

Let $(s_1, s_2 \neq 0)$ denote the position of an arbitrary source lying off the s_1-axis inside the caustic curve. Let (u_i, v_i) denote the corresponding lensed images. The total signed magnification μ at (s_1, s_2) is

$$\mu(u_i, v_i) = \frac{1}{8cu_1 - 4(u_1^2+v_1^2)} + \frac{1}{8cu_2 - 4(u_2^2+v_2^2)} + \frac{1}{8cu_3 - 4(u_3^2+v_3^2)} + \frac{1}{8cu_4 - 4(u_4^2+v_4^2)} \;. \tag{A3}$$

Our goal is to show that this sum is in fact identically zero. Let us begin by eliminating u from the lens equation (A1) to obtain a (depressed) quartic in v:

$$v^4 + (s_1 - 4c^2)v^2 + 2cs_2 v - \frac{s_2^2}{4} = 0\;. \tag{A4}$$

Knowing that this quartic must factor as

$$(v - v_1)(v - v_2)(v - v_3)(v - v_4) = 0\;, \tag{A5}$$

we expand (A5) and equate its coefficients to those of (A4). As a result we obtain four equations involving the v_i:

$$\mathbf{v_1 + v_2 + v_3 + v_4 \;=\; 0}\;, \tag{A6}$$

$$\mathbf{v_1v_2 + v_1v_3 + v_1v_4 + v_2v_3 + v_2v_4 + v_3v_4 \;=\; s_1 - 4c^2}\;, \tag{A7}$$

$$\mathbf{v_1v_2v_3 + v_1v_2v_4 + v_1v_3v_4 + v_2v_3v_4 \;=\; -2cs_2}\;, \tag{A8}$$

$$\mathbf{v_1v_2v_3v_4 \;=\; -\frac{s_2^2}{4}}\;. \tag{A9}$$

Next, we use (A1) to express each u_i in terms of v_i, bearing in mind that all $v_i \neq 0$:

$$u_i(v_i) = \frac{-s_2 + 4cv_i}{2v_i}\;. \tag{A10}$$

Our procedure is to insert (A10) into the total magnification (A3), thereby obtaining an expression involving only the v_i, $\mu = \mu(v_i)$, and to then simplify this expression to zero using (A6)–(A9).

Unfortunately, the equation $\mu = \mu(v_i)$, when written over a common denominator, is quite unwieldy. To simplify proceedings, we factor the numerator in terms of powers of s_2:

$$s_2^6\,\left(v_1^2 + v_2^2 + v_3^2 + v_4^2\right)\;, \tag{A11}$$

$$-4c\,s_2^5\,\left(v_1^2v_2+v_1v_2^2+v_1^2v_3+v_2^2v_3+v_1v_3^2+v_2v_3^2+v_1^2v_4+v_2^2v_4+v_3^2v_4+v_1v_4^2+v_2v_4^2+v_3v_4^2\right)\ , \tag{A12}$$

$$\begin{aligned} 4\,s_2^4\,\big((v_1^4v_2^2 &+ v_1^2v_2^4+v_1^4v_3^2+v_2^4v_3^2+v_1^2v_3^4+v_2^2v_3^4+v_1^4v_4^2+v_2^4v_4^2+v_3^4v_4^2+v_1^2v_4^4+v_2^2v_4^4+v_3^2v_4^4\big) \\ &+ 4c^2(v_1^2v_2v_3+v_1v_2^2v_3+v_1v_2v_3^2+v_1^2v_2v_4+v_1v_2^2v_4+v_1^2v_3v_4+v_2^2v_3v_4 \\ &+ v_1v_3^2v_4+v_2v_3^2v_4+v_1v_2v_4^2+v_1v_3v_4^2+v_2v_3v_4^2)\big)\ , \end{aligned} \tag{A13}$$

$$\begin{aligned} -16c\,s_2^3\,\big((v_1^4v_2^2v_3 &+ v_1^2v_2^4v_3+v_1^4v_2v_3^2+v_1v_2^4v_3^2+v_1^2v_2v_3^4+v_1v_2^2v_3^4+v_1^4v_2^2v_4+v_1^2v_2^4v_4+v_1^4v_3^2v_4+v_2^4v_3^2v_4 \\ &+ v_1^2v_3^4v_4+v_2^2v_3^4v_4+v_1^4v_2v_4^2+v_1v_2^4v_4^2+v_1^4v_3v_4^2+v_2^4v_3v_4^2+v_1v_3^4v_4^2+v_2v_3^4v_4^2 \\ &+ v_1^2v_2v_4^4+v_1v_2^2v_4^4+v_1^2v_3v_4^4+v_2^2v_3v_4^4+v_1v_3^2v_4^4+v_2v_3^2v_4^4\big) \\ &+ 4c^2\left(v_1^2v_2v_3v_4+v_1v_2^2v_3v_4+v_1v_2v_3^2v_4+v_1v_2v_3v_4^2\right)\big)\ , \end{aligned} \tag{A14}$$

$$\begin{aligned} 16\,s_2^2\,\big((v_1^4v_2^4v_3^2 &+ v_1^4v_2^2v_3^4+v_1^2v_2^4v_3^4+v_1^4v_2^4v_4^2+v_1^4v_3^4v_4^2+v_2^4v_3^4v_4^2+v_1^4v_2^2v_4^4+v_1^2v_2^4v_4^4+v_1^4v_3^2v_4^4+v_2^4v_3^2v_4^4 \\ &+ v_1^2v_3^4v_4^4+v_2^2v_3^4v_4^4\big)+4c^2\left(v_1^4v_2^2v_3v_4+v_1^2v_2^4v_3v_4+v_1^4v_2v_3^2v_4+v_1v_2^4v_3^2v_4+v_1^2v_2v_3^4v_4\right. \\ &+ \left.v_1v_2^2v_3^4v_4+v_1^4v_2v_3v_4^2+v_1v_2^4v_3v_4^2+v_1v_2v_3^4v_4^2+v_1^2v_2v_3v_4^4+v_1v_2^2v_3v_4^4+v_1v_2v_3^2v_4^4\right)\big)\ , \end{aligned} \tag{A15}$$

$$\begin{aligned} -64c\,s_2\,\big(v_1^4v_2^4v_3^2v_4 &+ v_1^4v_2^2v_3^4v_4+v_1^2v_2^4v_3^4v_4+v_1^4v_2^4v_3v_4^2+v_1^4v_2v_3^4v_4^2+v_1v_2^4v_3^4v_4^2+v_1^4v_2^2v_3v_4^4 \\ &+ v_1^2v_2^4v_3v_4^4+v_1^4v_2v_3^2v_4^4+v_1v_2^4v_3^2v_4^4+v_1^2v_2v_3^4v_4^4+v_1v_2^2v_3^4v_4^4\big)\ , \end{aligned} \tag{A16}$$

$$64\left(v_1^4v_2^4v_3^4v_4^2+v_1^4v_2^4v_3^2v_4^4+v_1^4v_2^2v_3^4v_4^4+v_1^2v_2^4v_3^4v_4^4\right)\ . \tag{A17}$$

When written over a common denominator, the numerator of $\mu=\mu(v_i)$ is therefore (A11) + (A12) + (A13) + (A14) + (A15) + (A16) + (A17). We now proceed to use (A6)–(A9) to simplify each of these terms, beginning with (A11), the s_2^6-term.

The s_2^6-term. We use (A6) and (A7):

$$\begin{aligned} (v_1+v_2+v_3+v_4)^2-2(v_1v_2+v_1v_3+v_1v_4+v_2v_3+v_2v_4+v_3v_4) &= v_1^2+v_2^2+v_3^2+v_4^2 \\ &= 0-2(s_1-4c^2)=8c^2-2s_1\ . \end{aligned} \tag{A18}$$

The s_2^6-term thus simplifies to

$$s_2^6(8c^2-2s_1)=\mathbf{8c^2s_2^6}-\mathbf{2s_1s_2^6}\ . \tag{A19}$$

Because now there is no v_i-dependence, this term has been fully simplified.

The s_2^5-term. We use (A6)–(A8):

$$\begin{aligned} (v_1 &+ v_2+v_3+v_4)\,(v_1v_2+v_1v_3+v_1v_4+v_2v_3+v_2v_4+v_3v_4)-3\,(v_1v_2v_3+v_1v_2v_4+v_1v_3v_4+v_2v_3v_4) \\ &= v_1^2v_2+v_1v_2^2+v_1^2v_3+v_2^2v_3+v_1v_3^2+v_2v_3^2+v_1^2v_4+v_2^2v_4+v_3^2v_4+v_1v_4^2+v_2v_4^2+v_3v_4^2 \\ &= 0\ (s_1-4c^2)-3(-2cs_2)=6cs_2\ . \end{aligned} \tag{A20}$$

The s_2^5-term thus simplifies to

$$-4cs_2^5(6cs_2)=-\mathbf{24c^2s_2^6}\ . \tag{A21}$$

The s_2^4-term. We proceed in steps. First,

$$\begin{aligned} (v_1 &+ v_2+v_3+v_4)\,(v_1v_2v_3+v_1v_2v_4+v_1v_3v_4+v_2v_3v_4)-4v_1v_2v_3v_4 \\ &= v_1^2v_2v_3+v_1v_2^2v_3+v_1v_2v_3^2+v_1^2v_2v_4+v_1v_2^2v_4+v_1^2v_3v_4+v_2^2v_3v_4+v_1v_3^2v_4 \\ &\qquad +v_2v_3^2v_4+v_1v_2v_4^2+v_1v_3v_4^2+v_2v_3v_4^2 \\ &= 0\ (-2cs_2)-4\left(-\frac{s_2^2}{4}\right)=s_2^2\ . \end{aligned} \tag{A22}$$

Second,

$$\begin{aligned}(v_1v_2 &+ v_1v_3+v_1v_4+v_2v_3+v_2v_4+v_3v_4)^2 - 2\left(v_1^2v_2v_3+v_1v_2^2v_3+v_1v_2v_3^2+v_1^2v_2v_4+v_1v_2^2v_4\right.\\ &+ \left. v_1^2v_3v_4+v_2^2v_3v_4+v_1v_3^2v_4+v_2v_3^2v_4+v_1v_2v_4^2+v_1v_3v_4^2+v_2v_3v_4^2\right) - 6v_1v_2v_3v_4\\ &= v_1^2v_2^2+v_1^2v_3^2+v_2^2v_3^2+v_1^2v_4^2+v_2^2v_4^2+v_3^2v_4^2\\ &= (s_1-4c^2)^2-2(s_2^2)-6\left(-\frac{s_2^2}{4}\right) = (s_1-4c^2)^2-\frac{s_2^2}{2}\ .\end{aligned} \tag{A23}$$

Third,

$$\begin{aligned}(v_1v_2v_3 &+ v_1v_2v_4+v_1v_3v_4+v_2v_3v_4)^2 - 2\,(v_1v_2+v_1v_3+v_1v_4+v_2v_3+v_2v_4+v_3v_4)(v_1v_2v_3v_4)\\ &= v_1^2v_2^2v_3^2+v_1^2v_2^2v_4^2+v_1^2v_3^2v_4^2+v_2^2v_3^2v_4^2\\ &= (-2cs_2)^2-2(s_1-4c^2)\left(-\frac{s_2^2}{4}\right) = 2c^2s_2^2+\frac{s_1s_2^2}{2}\ .\end{aligned} \tag{A24}$$

Fourth, we combine (A18), (A23), and (A24) as follows:

$$\begin{aligned}(v_1^2 &+ v_2^2+v_3^2+v_4^2)\left(v_1^2v_2^2+v_1^2v_3^2+v_2^2v_3^2+v_1^2v_4^2+v_2^2v_4^2+v_3^2v_4^2\right) - 3\left(v_1^2v_2^2v_3^2+v_1^2v_2^2v_4^2+v_1^2v_3^2v_4^2+v_2^2v_3^2v_4^2\right)\\ &= v_1^4v_2^2+v_1^2v_2^4+v_1^4v_3^2+v_2^4v_3^2+v_1^2v_3^4+v_2^2v_3^4+v_1^4v_4^2+v_2^4v_4^2+v_3^4v_4^2+v_1^2v_4^4+v_2^2v_4^4+v_3^2v_4^4\\ &= \left(8c^2-2s_1\right)\left((s_1-4c^2)^2-\frac{s_2^2}{2}\right)-3\left(2c^2s_2^2+\frac{s_1s_2^2}{2}\right)\\ &= 128c^6-96c^4s_1+24c^2s_1^2-2s_1^3-10c^2s_2^2-\frac{s_1s_2^2}{2}\ .\end{aligned} \tag{A25}$$

Finally, using (A25) and (A22), the s_2^4-term simplifies to

$$\begin{aligned}&4s_2^4\left(\left(128c^6-96c^4s_1+24c^2s_1^2-2s_1^3-10c^2s_2^2-\frac{s_1s_2^2}{2}\right)+4c^2(s_2^2)\right)\\ &\quad = \mathbf{512c^6s_2^4-384c^4s_1s_2^4+96c^2s_1^2s_2^4-8s_1^3s_2^4-24c^2s_2^6-2s_1s_2^6}\ .\end{aligned} \tag{A26}$$

The s_2^3-term. Once again we proceed in steps. First, we note that

$$4c^2\left(v_1^2v_2v_3v_4+v_1v_2^2v_3v_4+v_1v_2v_3^2v_4+v_1v_2v_3v_4^2\right) = 4c^2v_1v_2v_3v_4(v_1+v_2+v_3+v_4) = 0\ ,$$

so that the s_2^3-term reduces to

$$\begin{aligned}-16c\,s_2^3\,\left(v_1^4v_2^2v_3\right. &+ v_1^2v_2^4v_3+v_1^4v_2v_3^2+v_1v_2^4v_3^2+v_1^2v_2v_3^4+v_1v_2^2v_3^4+v_1^4v_2^2v_4+v_1^2v_2^4v_4+v_1^4v_3^2v_4+v_2^4v_3^2v_4\\ &+ v_1^2v_3^4v_4+v_2^2v_3^4v_4+v_1^4v_2v_4^2+v_1v_2^4v_4^2+v_1^4v_3v_4^2+v_2^4v_3v_4^2+v_1v_3^4v_4^2+v_2v_3^4v_4^2\\ &+ \left. v_1^2v_2v_4^4+v_1v_2^2v_4^4+v_1^2v_3v_4^4+v_2^2v_3v_4^4+v_1v_3^2v_4^4+v_2v_3^2v_4^4\right)\ .\end{aligned}$$

Second, we multiply our equation by $1 = \frac{(-s_2^2/4)}{v_1v_2v_3v_4}$, which is a valid operation since each $v_i \neq 0$, and group together terms with a common denominator to obtain

$$\begin{aligned}4c\,s_2^5\,&\left(\frac{v_3v_4^3+v_2^3v_3+v_2v_3^3+v_2^3v_4+v_3^3v_4+v_2v_4^3}{v_1}+\frac{v_1^3v_3+v_1v_3^3+v_1^3v_4+v_3^3v_4+v_1v_4^3+v_3v_4^3}{v_2}\right.\\ &\left.+\frac{v_1v_4^3+v_2v_4^3+v_1^3v_2+v_1v_2^3+v_1^3v_4+v_2^3v_4}{v_3}+\frac{v_1^3v_2+v_1v_2^3+v_1^3v_3+v_2^3v_3+v_1v_3^3+v_2v_3^3}{v_4}\right)\ .\end{aligned} \tag{A27}$$

Third, we use (A18), (A7), and (A22) to obtain

$$\begin{aligned}\left(v_1^2\right. &+ \left.v_2^2+v_3^2+v_4^2\right)(v_1v_2+v_1v_3+v_1v_4+v_2v_3+v_2v_4+v_3v_4) - \left(v_1^2v_2v_3+v_1v_2^2v_3+v_1v_2v_3^2\right.\\ &+ \left. v_1^2v_2v_4+v_1v_2^2v_4+v_1^2v_3v_4+v_2^2v_3v_4+v_1v_3^2v_4+v_2v_3^2v_4+v_1v_2v_4^2+v_1v_3v_4^2+v_2v_3v_4^2\right)\\ &= v_1^3v_2+v_1v_2^3+v_1^3v_3+v_2^3v_3+v_1v_3^3+v_2v_3^3+v_1^3v_4+v_2^3v_4+v_3^3v_4+v_1v_4^3+v_2v_4^3+v_3v_4^3\\ &= (8c^2-2s_1)(-4c^2+s_1)-s_2^2 = -2(4c^2-s_1)^2-s_2^2\ .\end{aligned} \tag{A28}$$

For each of the four terms in (A27), we use (A28) to simplify it. For example, the first term in (A27) simplifies as follows:

$$\begin{aligned}\frac{v_3v_4^3+v_2^3v_3+v_2v_3^3+v_2^3v_4+v_3^3v_4+v_2v_4^3}{v_1} &= \frac{-2(4c^2-s_1)^2-s_2^2-v_1^3v_2-v_1v_2^3-v_1^3v_3-v_1v_3^3-v_1^3v_4-v_1v_4^3}{v_1}\\ &= \frac{-2(4c^2-s_1)^2-s_2^2}{v_1}-v_1^2v_2-v_2^3-v_1^2v_3-v_3^3-v_1^2v_4-v_4^3\ .\end{aligned}$$

Likewise with the remaining terms in (A27), so that the s_2^3-term reduces to

$$\begin{aligned}4c\,s_2^5\,\Bigg(&\left(-2(4c^2-s_1)^2-s_2^2\right)\left(\frac{1}{v_1}+\frac{1}{v_2}+\frac{1}{v_3}+\frac{1}{v_4}\right)-v_1^2v_2-v_1^2v_3-v_1^2v_4-v_1v_2^2-v_2^2v_3\\ &-v_2^2v_4-v_1v_3^2-v_2v_3^2-v_3^2v_4-v_1v_4^2-v_2v_4^2-v_3v_4^2-3\left(v_1^3+v_2^3+v_2^3+v_2^3\right)\Bigg)\ .\end{aligned}\tag{A29}$$

Fourth, we use (A18), (A6), and (A20) as follows:

$$\begin{aligned}\left(v_1^2\right.&+\left.v_2^2+v_3^2+v_4^2\right)(v_1+v_2+v_3+v_4)\\ &-\left(v_1^2v_2+v_1v_2^2+v_1^2v_3+v_2^2v_3+v_1v_3^2+v_2v_3^2+v_1^2v_4+v_2^2v_4+v_3^2v_4+v_1v_4^2+v_2v_4^2+v_3v_4^2\right)\\ &= v_1^3+v_2^3+v_3^3+v_4^3=(8c^2-2s_1)\,0-6cs_2=-6cs_2\ .\end{aligned}\tag{A30}$$

Fifth, we use (A8) and (A9) to obtain

$$\frac{1}{v_1}+\frac{1}{v_2}+\frac{1}{v_3}+\frac{1}{v_4}=\frac{v_2v_3v_4+v_1v_3v_4+v_1v_2v_4+v_1v_2v_3}{v_1v_2v_3v_4}=\frac{8c}{s_2}\ .\tag{A31}$$

Finally, we insert (A30) and (A31) back into (A29) to obtain the simplified form of the s_2^3-term:

$$4c\,s_2^5\left(\left(-2(4c^2-s_1)^2-s_2^2\right)\left(\frac{8c}{s_2}\right)-(6cs_2)-3(-6cs_2)\right)=\mathbf{-1024c^6s_2^4+512c^4s_1s_2^4-64c^2s_1^2s_2^4+16c^2s_2^6}\ .\tag{A32}$$

The s_2^2-term: Let us begin with the portion of this term with no c-dependence, namely,

$$\begin{aligned}16\,s_2^2\,\big(v_1^4v_2^4v_3^2 &+ v_1^4v_2^2v_3^4+v_1^2v_2^4v_3^4+v_1^4v_2^4v_4^2+v_1^4v_3^4v_4^2+v_2^4v_3^4v_4^2+v_1^4v_2^2v_4^4+v_1^2v_2^4v_4^4\\ &+ v_1^4v_3^2v_4^4+v_2^4v_3^2v_4^4+v_1^2v_3^4v_4^4+v_2^2v_3^4v_4^4\big)\ .\end{aligned}\tag{A33}$$

Analogous to the s_2^3-term above, we begin by multiplying through twice by $1=\frac{(-s_2^2/4)}{v_1v_2v_3v_4}$, and then grouping together terms with a common denominator, to obtain

$$s_2^6\ \left(\frac{v_2^2v_3^2+v_2^2v_4^2+v_3^2v_4^2}{v_1^2}+\frac{v_1^2v_3^2+v_1^2v_4^2+v_3^2v_4^2}{v_2^2}+\frac{v_1^2v_4^2+v_2^2v_4^2+v_1^2v_2^2}{v_3^2}+\frac{v_1^2v_2^2+v_1^2v_3^2+v_2^2v_3^2}{v_4^2}\right)\ .$$

Next, we use (A23) on each of the four terms. For example, the first term simplifies as follows:

$$\begin{aligned}\frac{v_2^2v_3^2+v_2^2v_4^2+v_3^2v_4^2}{v_1^2} &= \frac{-v_1^2v_2^2-v_1^2v_3^2-v_1^2v_4^2+(-4c^2+s_1)^2-s_2^2/2}{v_1^2}\\ &= -v_2^2-v_3^2-v_4^2+\frac{(-4c^2+s_1)^2-s_2^2/2}{v_1^2}\ .\end{aligned}$$

Likewise with the remaining terms, so that (A33) reduces to

$$s_2^6\ \left(-3\left(v_1^2+v_2^2+v_3^2+v_4^2\right)+\left((-4c^2+s_1)^2-\frac{s_2^2}{2}\right)\left(\frac{1}{v_1^2}+\frac{1}{v_2^2}+\frac{1}{v_3^2}+\frac{1}{v_4^2}\right)\right)\ .$$

Next, we use (A24) and (A9) to obtain

$$\frac{1}{v_1^2}+\frac{1}{v_2^2}+\frac{1}{v_3^2}+\frac{1}{v_4^2}=\frac{v_1^2v_2^2v_3^2+v_1^2v_2^2v_4^2+v_1^2v_3^2v_4^2+v_2^2v_3^2v_4^2}{v_1^2v_2^2v_3^2v_4^2}=\frac{8(4c^2+s_1)}{s_2^2}\ ,$$

Using this and (A18), we see that (A33) simplifies to

$$\begin{aligned} & s_2^6 \left(-3\left(8c^2-2s_1\right)+\left((-4c^2+s_1)^2-\frac{s_2^2}{2}\right)\left(\frac{8(4c^2+s_1)}{s_2^2}\right)\right) \\ & = 512c^6s_2^4-128c^4s_1s_2^4-32c^2s_1^2s_2^4+8s_1^3s_2^4-40c^2s_2^6+2s_1s_2^6. \end{aligned} \tag{A34}$$

There remains the portion of the s_2^2-term with a factor of c^2, namely

$$\begin{aligned} 64c^2\,s_2^2\,\big(v_1^4v_2^2v_3v_4 \;&+\; v_1^2v_2^4v_3v_4+v_1^4v_2v_3^2v_4+v_1v_2^4v_3^2v_4+v_1^2v_2v_3^4v_4+v_1v_2^2v_3^4v_4 \\ &+\; v_1^4v_2v_3v_4^2+v_1v_2^4v_3v_4^2+v_1v_2v_3^4v_4^2+v_1^2v_2v_3v_4^4+v_1v_2^2v_3v_4^4+v_1v_2v_3^2v_4^4\big)\ , \end{aligned} \tag{A35}$$

which factors as

$$\begin{aligned} & 64c^2\,s_2^2\,v_1v_2v_3v_4\left(v_1^3v_2+v_1v_2^3+v_1^3v_3+v_2^3v_3+v_1v_3^3+v_2v_3^3+v_1^3v_4+v_2^3v_4+v_3^3v_4+v_1v_4^3+v_2v_4^3+v_3v_4^3\right) \\ & \quad = -16c^2\,s_2^4\,\left(v_1^3v_2+v_1v_2^3+v_1^3v_3+v_2^3v_3+v_1v_3^3+v_2v_3^3+v_1^3v_4+v_2^3v_4+v_3^3v_4+v_1v_4^3+v_2v_4^3+v_3v_4^3\right)\ . \end{aligned}$$

To further simplify this expression, we use (A18), (A7), and (A22):

$$\begin{aligned} \big(v_1^2+v_2^2 \;&+\; v_3^2+v_4^2\big)\left(v_1v_2+v_1v_3+v_1v_4+v_2v_3+v_2v_4+v_3v_4\right)-\big(v_1^2v_2v_3+v_1v_2^2v_3+v_1v_2v_3^2+v_1^2v_2v_4 \\ &+\; v_1v_2^2v_4+v_1^2v_3v_4+v_2^2v_3v_4+v_1v_3^2v_4+v_2v_3^2v_4+v_1v_2v_4^2+v_1v_3v_4^2+v_2v_3v_4^2\big) \\ &=\; v_1^3v_2+v_1v_2^3+v_1^3v_3+v_2^3v_3+v_1v_3^3+v_2v_3^3+v_1^3v_4+v_2^3v_4+v_3^3v_4+v_1v_4^3+v_2v_4^3+v_3v_4^3 \\ &=\; (8c^2-2s_1)(s_1-4c^2)-s_2^2=-32c^4+16c^2s_1-2s_1^2-s_2^2\ . \end{aligned}$$

Thus (A35) simplifies to

$$-16c^2s_2^4(-32c^4+16c^2s_1-2s_1^2-s_2^2)=512c^6s_2^4-256c^4s_1s_2^4+32c^2s_1^2s_2^4+16c^2s_2^6\ . \tag{A36}$$

Finally, we add (A34) and (A36) to obtain the simplified form of the s_2^2-term:

$$\mathbf{1024c^6s_2^4-384c^4s_1s_2^4+8s_1^3s_2^4-24c^2s_2^6+2s_1s_2^6}\ . \tag{A37}$$

The s_2-term. First, we factor it as

$$\begin{aligned} -64c\;\;s_2\;\;v_1v_2v_3v_4\,\big(&v_1^3v_2^3v_3+v_1^3v_2v_3^3+v_1v_2^3v_3^3+v_1^3v_2^3v_4+v_1^3v_3^3v_4+v_2^3v_3^3v_4+v_1^3v_2v_4^3+v_1v_2^3v_4^3 \\ &+v_1^3v_3v_4^3+v_2^3v_3v_4^3+v_1v_3^3v_4^3+v_2v_3^3v_4^3\big) \\ =\;\;16c\,s_2^3\,\big(&v_1^3v_2^3v_3+v_1^3v_2v_3^3+v_1v_2^3v_3^3+v_1^3v_2^3v_4+v_1^3v_3^3v_4+v_2^3v_3^3v_4+v_1^3v_2v_4^3+v_1v_2^3v_4^3 \\ &+v_1^3v_3v_4^3+v_2^3v_3v_4^3+v_1v_3^3v_4^3+v_2v_3^3v_4^3\big)\ . \end{aligned}$$

Second, we multiply our equation by $1=\frac{(-s_2^2/4)}{v_1v_2v_3v_4}$, group terms together with the same denominator, and use (A23) to obtain

$$\begin{aligned} 4c\,s_2^5\Bigg(v_1^2v_2 \;&+\; v_1v_2^2+v_1^2v_3+v_2^2v_3+v_1v_3^2+v_2v_3^2+v_1^2v_4+v_2^2v_4+v_3^2v_4+v_1v_4^2+v_2v_4^2+v_3v_4^2 \\ &-\;\left((-4c^2+s_1)^2-\frac{s_2^2}{2}\right)\left(\frac{1}{v_1}+\frac{1}{v_2}+\frac{1}{v_3}+\frac{1}{v_4}\right)\Bigg)\ . \end{aligned}$$

Finally, with the aid of (A20) and (A31), the s_2-term simplifies to

$$4c\,s_2^5\left(6cs_2-\left((-4c^2+s_1)^2-\frac{s_2^2}{2}\right)\left(\frac{8c}{s_2}\right)\right)=\mathbf{-512c^6s_2^4+256c^4s_1s_2^4-32c^2s_1^2s_2^4+40c^2s_2^6}\ . \tag{A38}$$

The term with no s_2-dependence. We factor this term as

$$64v_1^2v_2^2v_3^2v_4^2\left(v_1^2v_2^2v_3^2+v_1^2v_2^2v_4^2+v_1^2v_3^2v_4^2+v_2^2v_3^2v_4^2\right)$$

and then use (A9) and (A24) to obtain

$$64\left(-\frac{s_2^2}{4}\right)^2\left(2c^2s_2^2+\frac{s_1s_2^2}{2}\right)=\mathbf{8c^2s_2^6}+\mathbf{2s_1s_2^6}\ . \tag{A39}$$

We are done: for the simplified forms of (A11)–(A17), namely, equations (A19), (A21), (A26), (A32), (A37), (A38), and (A39), sum to zero:

$$\begin{aligned}
\text{(A19)} \;&+\; \text{(A21)}+\text{(A26)}+\text{(A32)}+\text{(A37)}+\text{(A38)}+\text{(A39)}\\
&=\; \left(8c^2s_2^6-2s_1s_2^6\right)+\left(-24c^2s_2^6\right)+\left(512c^6s_2^4-384c^4s_1s_2^4+96c^2s_1^2s_2^4-8s_1^3s_2^4-24c^2s_2^6-2s_1s_2^6\right)\\
&+\; \left(-1024c^6s_2^4+512c^4s_1s_2^4-64c^2s_1^2s_2^4+16c^2s_2^6\right)+\left(1024c^6s_2^4-384c^4s_1s_2^4+8s_1^3s_2^4-24c^2s_2^6+2s_1s_2^6\right)\\
&+\; \left(-512c^6s_2^4+256c^4s_1s_2^4-32c^2s_1^2s_2^4+40c^2s_2^6\right)+\left(8c^2s_2^6+2s_1s_2^6\right)=0\ .
\end{aligned}$$

This completes the proof for the quantitative form of the lensing map in the neighborhood of an elliptic umbilic. □

2. Hyperbolic Umbilic

The derivation of the quantitative form of the lensing map in the neighborhood of a hyperbolic umbilic critical point, can be found in [12, Chap. 6]. The resulting map is

$$\begin{aligned} s_1 &= u^2+2cv\ ,\\ s_2 &= v^2+2cu\ , \end{aligned} \tag{A40}$$

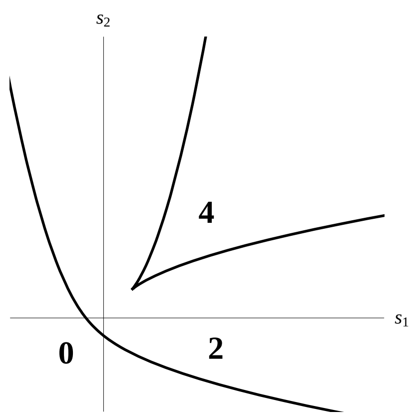

FIG. 4: Caustic curve for hyperbolic umbilic lensing map $\boldsymbol{\eta}_c$ in equation (A40) or Table I. The number of lensed images for sources in their respective regions is indicated.

and the corresponding magnification of an image (u, v) is

$$\left(\det(\text{Jac }\mathbf{s})\right)^{-1}(u,v) \;=\; \frac{1}{4(uv-c^2)}\ .$$

The critical curves are hyperbolas given by

$$v \;=\; \frac{c^2}{u}\ , \tag{A41}$$

and the corresponding caustic curve is

$$\begin{aligned} s_1 &= u^2+\frac{2c^3}{u}\ ,\\ s_2 &= 2cu+\frac{c^4}{u^2}\ . \end{aligned} \tag{A42}$$

The caustic curve is shown in Figure 4. The region “inside the beak” constitutes the four-image region. We now show that for all sources inside this region, the total signed magnification is identically zero.

First, note from the second lens equation (A40) that since $u_i(v_i) = \frac{s_2 - v_i^2}{2c}$, which is to say, since v_i does not appear in the denominator, we do not need to restrict our analysis to the case where $v_i \neq 0$, as we did with the elliptic umbilic. We will therefore let (s_1, s_2) denote an arbitrary source in the four-image region. The total signed magnification μ at (s_1, s_2) is

$$\mu(u_i, v_i) = \frac{1}{4(u_1 v_1 - c^2)} + \frac{1}{4(u_2 v_2 - c^2)} + \frac{1}{4(u_3 v_3 - c^2)} + \frac{1}{4(u_4 v_4 - c^2)} \ . \tag{A43}$$

We begin by eliminating u from the lens equation (A40) to obtain a (depressed) quartic in v:

$$v^4 - 2s_2 v^2 + 8c^3 v + (s_2^2 - 4c^2 s_1) = 0 \ .$$

Analogous to (A6)–(A9) and (B6)–(B9), we obtain four equations involving only the v_i:

$$\mathbf{v_1 + v_2 + v_3 + v_4} \ = \ \mathbf{0} \ , \tag{A44}$$
$$\mathbf{v_1 v_2 + v_1 v_3 + v_1 v_4 + v_2 v_3 + v_2 v_4 + v_3 v_4} \ = \ \mathbf{-2s_2} \ , \tag{A45}$$
$$\mathbf{v_1 v_2 v_3 + v_1 v_2 v_4 + v_1 v_3 v_4 + v_2 v_3 v_4} \ = \ \mathbf{-8c^3} \ , \tag{A46}$$
$$\mathbf{v_1 v_2 v_3 v_4} \ = \ \mathbf{-4c^2 s_1 + s_2^2} \ . \tag{A47}$$

We then insert $u_i(v_i)$ into the total magnification (A43) and factor the numerator of the resulting expression in powers of s_2:

$$c\, s_2^3 \left(v_1 v_2 v_3 + v_1 v_2 v_4 + v_1 v_3 v_4 + v_2 v_3 v_4\right) \ , \tag{A48}$$

$$\begin{aligned} - c\, s_2^2 \ \big(4c^3 \,(v_1 v_2 \ &+ \ v_1 v_3 + v_2 v_3 + v_1 v_4 + v_2 v_4 + v_3 v_4) + \big(v_1^3 v_2 v_4 + v_1 v_2^3 v_4 + v_1^3 v_3 v_4 + v_2^3 v_3 v_4 \\ &+ \ v_1 v_3^3 v_4 + v_2 v_3^3 v_4 + v_1 v_2 v_4^3 + v_1 v_3 v_4^3 + v_2 v_3 v_4^3 + v_1^3 v_2 v_3 + v_1 v_2^3 v_3 + v_1 v_2 v_3^3\big)\big) \ , \end{aligned} \tag{A49}$$

$$\begin{aligned} c\, s_2 \ \big(12c^6 \,(v_1 \ &+ \ v_2 + v_3 + v_4) + 4c^3 \left(v_1^3 v_2 + v_1 v_2^3 + v_1^3 v_3 + v_2^3 v_3 + v_1 v_3^3 + v_2 v_3^3 + v_1^3 v_4 + v_2^3 v_4 + v_3^3 v_4\right. \\ &+ \ \left. v_1 v_4^3 + v_2 v_4^3 + v_3 v_4^3\right) + \left(v_1^3 v_2^3 v_3 + v_1^3 v_2 v_3^3 + v_1 v_2^3 v_3^3 + v_1^3 v_2^3 v_4 + v_1^3 v_3^3 v_4\right. \\ &+ \ \left. v_2^3 v_3^3 v_4 + v_1^3 v_2 v_4^3 + v_1 v_2^3 v_4^3 + v_1^3 v_3 v_4^3 + v_2^3 v_3 v_4^3 + v_1 v_3^3 v_4^3 + v_2 v_3^3 v_4^3\right)\big) \ , \end{aligned} \tag{A50}$$

$$\begin{aligned} - 32c^{10} \ &- \ 12c^7 \left(v_1^3 + v_2^3 + v_4^3 + v_3^3\right) - 4c^4 \left(v_1^3 v_3^3 + v_2^3 v_3^3 + v_1^3 v_2^3 + v_1^3 v_4^3 + v_2^3 v_4^3 + v_3^3 v_4^3\right) \\ &- \ c \left(v_1^3 v_2^3 v_4^3 + v_1^3 v_3^3 v_4^3 + v_2^3 v_3^3 v_4^3 + v_1^3 v_2^3 v_3^3\right) \ . \end{aligned} \tag{A51}$$

The s_2^3-term. Using (A46), this term simplifies to

$$c s_2^3 (-2 s_2) = \mathbf{-8c^4 s_2^3} \ . \tag{A52}$$

The s_2^2-term. Analogous to (A18), we have $v_1^2 + v_2^2 + v_3^2 + v_4^2 = 4s_2$, which, when combined with (A44), (A46), and (A47), yields

$$\begin{aligned} \big(v_1^2 \ &+ \ v_2^2 + v_3^2 + v_4^2\big) \left(v_1 v_2 v_3 + v_1 v_2 v_4 + v_1 v_3 v_4 + v_2 v_3 v_4\right) - (v_1 v_2 v_3 v_4)(v_1 + v_2 + v_3 + v_4) \\ &= \ v_1^3 v_2 v_4 + v_1 v_2^3 v_4 + v_1^3 v_3 v_4 + v_2^3 v_3 v_4 + v_1 v_3^3 v_4 + v_2 v_3^3 v_4 + v_1 v_2 v_4^3 + v_1 v_3 v_4^3 + v_2 v_3 v_4^3 \\ &+ \ v_1^3 v_2 v_3 + v_1 v_2^3 v_3 + v_1 v_2 v_3^3 = (4s_2)(-8c^3) - (-4c^2 s_1)\, 0 = -32c^3 s_2 \ . \end{aligned}$$

The s_2^2-term thus simplifies to

$$- c \ s_2^2 \left(4c^3\,(-2s_2) - 32c^3 s_2\right) = \mathbf{40c^4 s_2^3} \ . \tag{A53}$$

The s_2-term. We proceed in steps. First, analogous to (A22), we have

$$\begin{aligned} v_1^2 v_2 v_3 \ &+ \ v_1 v_2^2 v_3 + v_1 v_2 v_3^2 + v_1^2 v_2 v_4 + v_1 v_2^2 v_4 + v_1^2 v_3 v_4 + v_2^2 v_3 v_4 + v_1 v_3^2 v_4 \\ &+ \ v_2 v_3^2 v_4 + v_1 v_2 v_4^2 + v_1 v_3 v_4^2 + v_2 v_3 v_4^2 \\ &= \ -4(-4c^2 s_1 + s_2^2) = 16c^2 s_1 - 4s_2^2 \ , \end{aligned}$$

which, as with (A28) above, is used to obtain

$$\begin{aligned}\left(v_1^2\right. &+& v_2^2+v_3^2+\left.v_4^2\right)\left(v_1v_2+v_1v_3+v_1v_4+v_2v_3+v_2v_4+v_3v_4\right)-\left(v_1^2v_2v_3+v_1v_2^2v_3+v_1v_2v_3^2+v_1^2v_2v_4\right.\\ &+& \left.v_1v_2^2v_4+v_1^2v_3v_4+v_2^2v_3v_4+v_1v_3^2v_4+v_2v_3^2v_4+v_1v_2v_4^2+v_1v_3v_4^2+v_2v_3v_4^2\right)\\ &=& v_1^3v_2+v_1v_2^3+v_1^3v_3+v_2^3v_3+v_1v_3^3+v_2v_3^3+v_1^3v_4+v_2^3v_4+v_3^3v_4+v_1v_4^3+v_2v_4^3+v_3v_4^3\\ &=& (4s_2)(-2s_2)-(-4s_2^2+16c^2s_1)=-4s_2^2-16c^2s_1\ . \end{aligned} \tag{A54}$$

Second, analogous to (A20) and (A23), we have

$$v_1^2v_2+v_1v_2^2+v_1^2v_3+v_2^2v_3+v_1v_3^2+v_2v_3^2+v_1^2v_4+v_2^2v_4+v_3^2v_4+v_1v_4^2+v_2v_4^2+v_3v_4^2=-3(-8c^3)=24c^3 \tag{A55}$$

and

$$v_1^2v_2^2+v_1^2v_3^2+v_2^2v_3^2+v_1^2v_4^2+v_2^2v_4^2+v_3^2v_4^2=4s_2^2+2(-4c^2s_1+s_2^2)=6s_2^2-8c^2s_1\ .$$

Third, we use (A47) and (A55) to obtain

$$\begin{aligned} v_1v_2v_3v_4\left(v_1^2v_2\right. &+& \left.v_1v_2^2+v_1^2v_3+v_2^2v_3+v_1v_3^2+v_2v_3^2+v_1^2v_4+v_2^2v_4+v_3^2v_4+v_1v_4^2+v_2v_4^2+v_3v_4^2\right)\\ &=& v_1^3v_2^2v_3v_4+v_1^2v_2^3v_3v_4+v_1^3v_2v_3^2v_4+v_1v_2^3v_3^2v_4+v_1^2v_2v_3^3v_4+v_1v_2^2v_3^3v_4\\ &+& v_1^3v_2v_3v_4^2+v_1v_2^3v_3v_4^2+v_1v_2v_3^3v_4^2+v_1^2v_2v_3v_4^3+v_1v_2^2v_3v_4^3+v_1v_2v_3^2v_4^3\\ &=& (s_2^2-4c^2s_1)24c^3\ . \end{aligned}$$

Fourth, we combine these three equations to obtain

$$\begin{aligned}\left(v_1^2v_2^2\right. &+& \left.v_1^2v_3^2+v_2^2v_3^2+v_1^2v_4^2+v_2^2v_4^2+v_3^2v_4^2\right)\left(v_1v_2v_3+v_1v_2v_4+v_1v_3v_4+v_2v_3v_4\right)-\left(v_1^3v_2^2v_3v_4+v_1^2v_2^3v_3v_4+v_1^3v_2v_3^2v_4\right.\\ &+& \left.v_1v_2^3v_3^2v_4+v_1^2v_2v_3^3v_4+v_1v_2^2v_3^3v_4+v_1^3v_2v_3v_4^2+v_1v_2^3v_3v_4^2+v_1v_2v_3^3v_4^2+v_1^2v_2v_3v_4^3+v_1v_2^2v_3v_4^3+v_1v_2v_3^2v_4^3\right)\\ &=& v_1^3v_2^3v_3+v_1^3v_2v_3^3+v_1v_2^3v_3^3+v_1^3v_2^3v_4+v_1^3v_3^3v_4+v_2^3v_3^3v_4+v_1^3v_2v_4^3+v_1v_2^3v_4^3+v_1^3v_3v_4^3+v_2^3v_3v_4^3+v_1v_3^3v_4^3+v_2v_3^3v_4^3\\ &=& (6s_2^2-8c^2s_1)(-8c^3)-(s_2^2-4c^2s_1)24c^3=160c^5s_1-72c^3s_2^2\ . \end{aligned} \tag{A56}$$

Finally, we use (A47), (A54), and (A56) to simplify the s_2-term:

$$c\,s_2\ \left(12c^6(0)+4c^3\left(-4s_2^2-16c^2s_1\right)+\left(160c^5s_1-72c^3s_2^2\right)\right)=\mathbf{96c^6s_1s_2-88c^4s_2^3}\ . \tag{A57}$$

The term with no s_2-dependence. Again, we proceed in steps. First, analogous to (A24) and (A30), we have

$$v_1^2v_2^2v_3^2+v_1^2v_2^2v_4^2+v_1^2v_3^2v_4^2+v_2^2v_3^2v_4^2=(-8c^3)^2-2(s_2^2-4c^2s_1)(-2s_2)=64c^6+4s_2(-4c^2s_1+s_2^2)$$

and

$$v_1^3+v_2^3+v_3^3+v_4^3=-24c^3\ . \tag{A58}$$

Second, we have

$$\begin{aligned}\left(v_1^2v_2v_3\right. &+& v_1v_2^2v_3+v_1v_2v_3^2+v_1^2v_2v_4+v_1v_2^2v_4+v_1^2v_3v_4+v_2^2v_3v_4+v_1v_3^2v_4+v_2v_3^2v_4+v_1v_2v_4^2+v_1v_3v_4^2\\ &+& \left.v_2v_3v_4^2\right)\left(v_1v_2+v_1v_3+v_2v_3+v_1v_4+v_2v_4+v_3v_4\right)-3\left(v_1^2v_2^2v_3^2+v_1^2v_2^2v_4^2+v_1^2v_3^2v_4^2+v_2^2v_3^2v_4^2\right)\\ &-& 3v_1v_2v_3v_4\left(v_1^2+v_2^2+v_3^2+v_4^2\right)-3v_1v_2v_3v_4(v_1v_2+v_1v_3+v_2v_3+v_1v_4+v_2v_4+v_3v_4)\\ &=& v_1^3v_2^2v_3+v_1^2v_2^3v_3+v_1^3v_2v_3^2+v_1v_2^3v_3^2+v_1^2v_2v_3^3+v_1v_2^2v_3^3+v_1^3v_2^2v_4+v_1^2v_2^3v_4+v_1^2v_2^2v_3v_4\\ &+& v_1^3v_3^2v_4+v_1^2v_2v_3^2v_4+v_1v_2^2v_3^2v_4+v_2^3v_3^2v_4+v_1^2v_3^3v_4+v_2^2v_3^3v_4+v_1^3v_2v_4^2+v_1v_2^3v_4^2\\ &+& v_1^3v_3v_4^2+v_1^2v_2v_3v_4^2+v_1v_2^2v_3v_4^2+v_2^3v_3v_4^2+v_1v_2v_3^2v_4^2+v_1v_3^3v_4^2+v_2v_3^3v_4^2\\ &+& v_1^2v_2v_4^3+v_1v_2^2v_4^3+v_1^2v_3v_4^3+v_2^2v_3v_4^3+v_1v_3^2v_4^3+v_2v_3^2v_4^3\\ &=& (-4s_2^2+16c^2s_1)(-2s_2)-3\left(64c^6+4s_2(-4c^2s_1+s_2^2)\right)-3(s_2^2-4c^2s_1)(4s_2)-3(s_2^2-4c^2s_1)(-2s_2)\\ &=& -192c^6+40c^2s_1s_2-10s_2^3\ , \end{aligned}$$

which we use to obtain

$$\begin{aligned}
\left(v_1^2v_2^2\right. &+ \left. v_1^2v_3^2+v_2^2v_3^2+v_1^2v_4^2+v_2^2v_4^2+v_3^2v_4^2\right)\left(v_1v_2+v_1v_3+v_2v_3+v_1v_4+v_2v_4+v_3v_4\right) \\
&- \left(v_1^3v_2^2v_3+v_1^2v_2^3v_3+v_1^3v_2v_3^2+v_1v_2^3v_3^2+v_1^2v_2v_3^3+v_1v_2^2v_3^3+v_1^3v_2^2v_4+v_1^2v_2^3v_4+v_1^2v_2^2v_3v_4\right. \\
&+ v_1^3v_3^2v_4+v_1^2v_2v_3^2v_4+v_1v_2^2v_3^2v_4+v_2^3v_3^2v_4+v_1^2v_3^3v_4+v_2^2v_3^3v_4+v_1^3v_2v_4^2+v_1v_2^3v_4^2 \\
&+ v_1^3v_3v_4^2+v_1^2v_2v_3v_4^2+v_1v_2^2v_3v_4^2+v_2^3v_3v_4^2+v_1v_2v_3^2v_4^2+v_1v_3^3v_4^2+v_2v_3^3v_4^2 \\
&+ \left. v_1^2v_2v_4^3+v_1v_2^2v_4^3+v_1^2v_3v_4^3+v_2^2v_3v_4^3+v_1v_3^2v_4^3+v_2v_3^2v_4^3\right) \\
&= v_1^3v_2^3+v_1^3v_3^3+v_2^3v_3^3+v_1^3v_4^3+v_2^3v_4^3+v_3^3v_4^3 \\
&= \left(4s_2^2+2\left(-4c^2s_1+s_2^2\right)\right)(-2s_2)-\left(-192c^6+40c^2s_1s_2-10s_2^3\right) \\
&= 192c^6-24c^2s_1s_2-2s_2^3\ . \qquad \text{(A59)}
\end{aligned}$$

Third, we have

$$\begin{aligned}
& (v_1v_2v_3+v_1v_2v_4+v_1v_3v_4+v_2v_3v_4)(v_1v_2+v_1v_3+v_1v_4+v_2v_3+v_2v_4+v_3v_4)-3(v_1+v_2+v_3+v_4)(v_1v_2v_3v_4) \\
=\ & v_1^2v_2^2v_3+v_1^2v_2v_3^2+v_1v_2^2v_3^2+v_1^2v_2^2v_4+v_1^2v_3^2v_4+v_2^2v_3^2v_4+v_1^2v_2v_4^2+v_1v_2^2v_4^2+v_1^2v_3v_4^2+v_2^2v_3v_4^2+v_1v_3^2v_4^2+v_2v_3^2v_4^2 \\
=\ & (-8c^3)(-2s_2)=16c^3s_2\ ,
\end{aligned}$$

which we use to obtain

$$\begin{aligned}
\left(v_1^2v_2^2v_3^2\right. &+ \left. v_1^2v_2^2v_4^2+v_1^2v_3^2v_4^2+v_2^2v_3^2v_4^2\right)\left(v_1v_2v_3+v_1v_2v_4+v_1v_3v_4+v_2v_3v_4\right)-v_1v_2v_3v_4\left(v_1^2v_2^2v_3+v_1^2v_2v_3^2+v_1v_2^2v_3^2\right. \\
&+ \left. v_1^2v_2^2v_4+v_1^2v_3^2v_4+v_2^2v_3^2v_4+v_1^2v_2v_4^2+v_1v_2^2v_4^2+v_1^2v_3v_4^2+v_2^2v_3v_4^2+v_1v_3^2v_4^2+v_2v_3^2v_4^2\right) \\
&= v_1^3v_2^3v_3^3+v_1^3v_2^3v_4^3+v_1^3v_3^3v_4^3+v_2^3v_3^3v_4^3 \\
&= \left(64c^6+4s_2\left(-4c^2s_1+s_2^2\right)\right)(-8c^3)-(-4c^2s_1+s_2^2)(16c^3s_2) \\
&= -512c^9+192c^5s_1s_2-48c^3s_2^3\ . \qquad \text{(A60)}
\end{aligned}$$

Finally, we use (A58), (A59), and (A60) to simplify the term with no s_2-dependence:

$$\begin{aligned}
&- \ 32c^{10}-12c^7(-24^3)-4c^4\left(192c^6-24c^2s_1s_2-2s_2^3\right)-c\left(-512c^9+192c^5s_1s_2-48c^3s_2^3\right) \\
&= \ \mathbf{-96c^6s_1s_2+56c^4s_2^3}\ . \qquad \text{(A61)}
\end{aligned}$$

We can now verify that the simplified forms of (A48)–(A51), namely, equations (A52), (A53), (A57), and (A61), sum to zero:

$$\text{(A52)}+\text{(A53)}+\text{(A57)}+\text{(A61)}=\left(-8c^4s_2^3\right)+\left(40c^4s_2^3\right)+\left(96c^6s_1s_2-88c^4s_2^3\right)+\left(-96c^6s_1s_2+56c^4s_2^3\right)=0\ .$$

This completes the proof for the quantitative form of the lensing map in the neighborhood of a hyperbolic umbilic. □

APPENDIX B: PROOF OF THE MAIN THEOREM FOR GENERIC MAPPINGS

1. Elliptic Umbilic

The derivation of the generic form of a one-parameter family of maps between planes in the neighborhood of an elliptic umbilic critical point, can be found in [18, Chap. 8]. The resulting map is

$$\begin{aligned}
s_1 &= 3v^2-3u^2-2cu\ , \\
s_2 &= 6uv-2cv\ , \qquad \text{(B1)}
\end{aligned}$$

and the corresponding magnification of an image (u,v) is

$$\left(\det(\mathrm{Jac}\,\mathbf{s})\right)^{-1}(u,v)\ =\ \frac{1}{4c^2-36(u^2+v^2)}\ . \qquad \text{(B2)}$$

(Note that our notation differs from that of [18].) The caustic curve is shown in Figure 5. Although these equations are noticeably different from their lensing map analogues (A1) and (A2), it is nonetheless true that the total signed

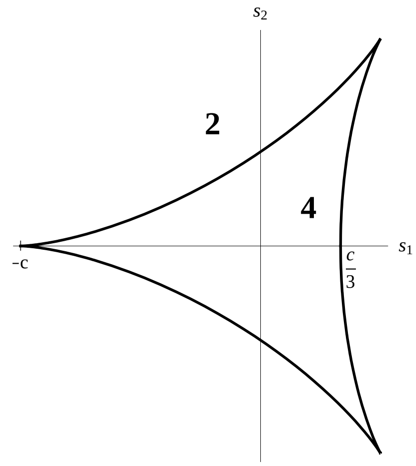

FIG. 5: Caustic curve for the generic elliptic umbilic mapping $\mathbf{f}_c$ in equation (B1) or Table I. The number of lensed images for sources in their respective regions is indicated.

magnification for quadruply-imaged sources is zero in this the generic case as well. The proof, in fact, is virtually identical to the lensing map case in Appendix A 1, as we now show.

As before, we begin by considering the special case of a source in the four-image region with $s_2 = 0$. In this case the generic lens equation (B1) is solvable. The lensed images are $\left(\frac{1}{3}\left(-c \pm \sqrt{c^2 - 3s_1}\,\right), 0\right)$, $\left(\frac{c}{3}, \pm\sqrt{\frac{c^2+s_1}{3}}\right)$, all of which are real because $-c < s_1 < \frac{c}{3}$ inside the caustic curve (see Figure 5). The total signed magnification, obtained by inserting each of these four images into (B2) and summing over, will be zero. Once again, therefore, we will restrict ourselves to sources (s_1, s_2) inside the caustic curve with $s_2 \neq 0$. Note from the second lens equation (B1) that $s_2 \neq 0$ forces the v-coordinate of each lensed image to be nonzero: $v_i \neq 0$.

Let $(s_1, s_2 \neq 0)$ denote the position of an arbitrary source lying off the s_1-axis inside the caustic curve. Let (u_i, v_i) once again denote the corresponding lensed images. The total signed magnification μ at (s_1, s_2) is

$$\mu(u_i, v_i) = \frac{1}{4c^2 - 36(u_1^2 + v_1^2)} + \frac{1}{4c^2 - 36(u_2^2 + v_2^2)} + \frac{1}{4c^2 - 36(u_3^2 + v_3^2)} + \frac{1}{4c^2 - 36(u_4^2 + v_4^2)} \,. \tag{B3}$$

We begin by eliminating u from the lens equation (B1) to obtain a (depressed) quartic in v:

$$v^4 - \frac{1}{3}(s_1 + c^2)v^2 - \frac{2}{9}cs_2 v - \frac{s_2^2}{36} = 0 \,. \tag{B4}$$

Knowing that this quartic must factor as

$$(v - v_1)(v - v_2)(v - v_3)(v - v_4) = 0 \,, \tag{B5}$$

we expand (B5) and equate its coefficients to those of (B4). As a result we obtain four equations involving the v_i:

$$\mathbf{v_1 + v_2 + v_3 + v_4 \;=\; 0} \,, \tag{B6}$$

$$\mathbf{v_1v_2 + v_1v_3 + v_1v_4 + v_2v_3 + v_2v_4 + v_3v_4 \;=\; -\frac{1}{3}(s_1 + c^2)} \,, \tag{B7}$$

$$\mathbf{v_1v_2v_3 + v_1v_2v_4 + v_1v_3v_4 + v_2v_3v_4 \;=\; \frac{2}{9}cs_2} \,, \tag{B8}$$

$$\mathbf{v_1v_2v_3v_4 \;=\; -\frac{s_2^2}{36}} \,. \tag{B9}$$

Next, we use the second equation in (B1) to express each u_i in terms of v_i, bearing in mind that all $v_i \neq 0$,

$$u_i(v_i) = \frac{s_2 + 2cv_i}{6v_i} \,. \tag{B10}$$

Once again, our procedure will be to insert (B10) into the total magnification (B3), thereby obtaining an expression involving only the v_i, $\mu = \mu(v_i)$. When we do so, and factor according to powers of s_2, we obtain

$$- s_2^6\ \left(v_1^2+v_2^2+v_3^2+v_4^2\right)\ ,$$

$$- 4\,c\,s_2^5\ \left(v_1^2v_2+v_1v_2^2+v_1^2v_3+v_2^2v_3+v_1v_3^2+v_2v_3^2+v_1^2v_4+v_2^2v_4+v_3^2v_4+v_1v_4^2+v_2v_4^2+v_3v_4^2\right)\ ,$$

$$\begin{aligned} - 4\,s_2^4\ \Big(9\big(v_1^4v_2^2 \ &+\ v_1^2v_2^4+v_1^4v_3^2+v_2^4v_3^2+v_1^2v_3^4+v_2^2v_3^4+v_1^4v_4^2+v_2^4v_4^2+v_3^4v_4^2+v_1^2v_4^4+v_2^2v_4^4+v_3^2v_4^4\big) \\ &+\ 4c^2\big(v_1^2v_2v_3+v_1v_2^2v_3+v_1v_2v_3^2+v_1^2v_2v_4+v_1v_2^2v_4+v_1^2v_3v_4+v_2^2v_3v_4 \\ &+\ v_1v_3^2v_4+v_2v_3^2v_4+v_1v_2v_4^2+v_1v_3v_4^2+v_2v_3v_4^2\big)\Big)\ , \end{aligned}$$

$$\begin{aligned} - 16\,c\,s_2^3\ \Big(9\big(v_1^4v_2^2v_3 \ &+\ v_1^2v_2^4v_3+v_1^4v_2v_3^2+v_1v_2^4v_3^2+v_1^2v_2v_3^4+v_1v_2^2v_3^4+v_1^4v_2^2v_4+v_1^2v_2^4v_4+v_1^4v_3^2v_4+v_2^4v_3^2v_4 \\ &+\ v_1^2v_3^4v_4+v_2^2v_3^4v_4+v_1^4v_2v_4^2+v_1v_2^4v_4^2+v_1^4v_3v_4^2+v_2^4v_3v_4^2+v_1v_3^4v_4^2+v_2v_3^4v_4^2 \\ &+\ v_1^2v_2v_4^4+v_1v_2^2v_4^4+v_1^2v_3v_4^4+v_2^2v_3v_4^4+v_1v_3^2v_4^4+v_2v_3^2v_4^4\big) \\ &-\ 4c^2\ \big(v_1^2v_2v_3v_4+4v_1v_2^2v_3v_4+4v_1v_2v_3^2v_4+4v_1v_2v_3v_4^2\big)\Big)\ , \end{aligned}$$

$$\begin{aligned} - 144\,s_2^2\ \Big(9\big(v_1^4v_2^4v_3^2 \ &+\ v_1^4v_2^2v_3^4+v_1^2v_2^4v_3^4+v_1^4v_2^4v_4^2+v_1^4v_3^4v_4^2+v_2^4v_3^4v_4^2+v_1^4v_2^2v_4^4+v_1^2v_2^4v_4^4+v_1^4v_3^2v_4^4+v_2^4v_3^2v_4^4 \\ &+\ v_1^2v_3^4v_4^4+v_2^2v_3^4v_4^4\big)+4c^2\big(v_1^4v_2^2v_3v_4+v_1^2v_2^4v_3v_4+v_1^4v_2v_3^2v_4+v_1v_2^4v_3^2v_4+v_1^2v_2v_3^4v_4 \\ &+\ v_1v_2^2v_3^4v_4+v_1^4v_2v_3v_4^2+v_1v_2^4v_3v_4^2+v_1v_2v_3^4v_4^2+v_1^2v_2v_3v_4^4+v_1v_2^2v_3v_4^4+v_1v_2v_3^2v_4^4\big)\Big)\ , \end{aligned}$$

$$\begin{aligned} - 5184\,c\,s_2\ \big(v_1^4v_2^4v_3^2v_4 \ &+\ v_1^4v_2^2v_3^4v_4+v_1^2v_2^4v_3^4v_4+v_1^4v_2^4v_3v_4^2+v_1^4v_2v_3^4v_4^2+v_1v_2^4v_3^4v_4^2+v_1^4v_2^2v_3v_4^4 \\ &+\ v_1^2v_2^4v_3v_4^4+v_1^4v_2v_3^2v_4^4+v_1v_2^4v_3^2v_4^4+v_1^2v_2v_3^4v_4^4+v_1v_2^2v_3^4v_4^4\big)\ , \end{aligned}$$

$$- 46656\left(v_1^4v_2^4v_3^4v_4^2+v_1^4v_2^4v_3^2v_4^4+v_1^4v_2^2v_3^4v_4^4+v_1^2v_2^4v_3^4v_4^4\right)\ .$$

Perusal of these expressions shows that aside from constant factors, the form of each of the expressions in parentheses is *identical* to its counterpart in the lensing map case in Appendix A 1. This means that the procedures we employed above, (A18)–(A39), carry through without modification. Of course, we expect different final answers, since the right-hand sides of (B7)–(B9) differ from those of (A7)–(A9). With that said, we can forgo the labor and only state the final results:

The s_2^6-term: $-\frac{2}{3}c^2s_2^6 - \frac{2}{3}s_1s_2^6$,

The s_2^5-term: $\frac{8}{3}c^2s_2^6$,

The s_2^4-term: $-\frac{8}{3}c^6s_2^4 - 8c^4s_1s_2^4 - 8c^2s_1^2s_2^4 - \frac{8}{3}s_1^3s_2^4 + \frac{26}{9}c^2s_2^6 - \frac{2}{9}s_1s_2^6$,

The s_2^3-term: $\frac{64}{9}c^6s_2^4 + \frac{128}{9}c^4s_1s_2^4 + \frac{65}{9}c^2s_1^2s_2^4 - \frac{16}{9}c^2s_2^6$,

The s_2^2-term: $-8c^6s_2^4 - \frac{40}{3}c^4s_1s_2^4 - \frac{8}{3}c^2s_1^2s_2^4 + \frac{8}{3}s_1^3s_2^4 + \frac{22}{9}c^2s_2^6 + \frac{2}{3}s_1s_2^6$,

The s_2-term: $\frac{32}{9}c^6s_2^4 + \frac{64}{9}c^4s_1s_2^4 + \frac{32}{9}c^2s_1^2s_2^4 - \frac{40}{9}c^2s_2^6$,

The term with no s_2-dependence: $-\frac{10}{9}c^2s_2^6 + \frac{2}{3}s_1s_2^6$.

We can now verify that these terms sum to zero. This completes the proof for the generic form of a one-parameter family of maps between planes in the neighborhood of an elliptic umbilic. $\square$

2. Hyperbolic Umbilic

The derivation of the generic form of a one-parameter family of maps between planes in the neighborhood of a hyperbolic umbilic critical point, can be found in [18, Chap. 8]. The resulting map is

$$\begin{aligned} s_1 &= -3u^2 - cv \ , \\ s_2 &= -3v^2 - cu \ , \end{aligned} \tag{B11}$$

and the corresponding magnification of an image (u, v) is

$$\left(\det(\mathrm{Jac}\ \mathbf{s})\right)^{-1}(u, v) \ = \ \frac{1}{-c^2 + 36uv} \ . \tag{B12}$$

The caustic curve is shown in Figure 6. We will show that the total signed magnification for quadruply-imaged sources is zero in this the generic case as well. What is more, just as the proof for the generic elliptic umbilic was virtually identical to its lensing map analogue in Appendix A 1, so will be the case here.

First, note from the second lens equation (B11) that since $u_i(v_i) = \frac{-s_2 - 3v_i^2}{c}$, which is to say, since v_i does not appear in the denominator, we do not need to restrict our analysis to the case where $v_i \neq 0$. Once again, therefore, we will let (s_1, s_2) denote an arbitrary source in the four-image region. The total signed magnification μ at (s_1, s_2) is

$$\mu(u_i, v_i) = \frac{1}{-c^2 + 36u_1v_1} + \frac{1}{-c^2 + 36u_2v_2} + \frac{1}{-c^2 + 36u_3v_3} + \frac{1}{-c^2 + 36u_4v_4} \ . \tag{B13}$$

This time the (depressed) quartic in v is

$$v^4 + \frac{2}{3}s_2v^2 + \frac{c^3}{27}v + \frac{3s_2^2 + c^2s_1}{27} = 0$$

and the corresponding four equations involving only the v_i are

$$\mathbf{v_1 + v_2 + v_3 + v_4 \ = \ 0} \ , \tag{B14}$$

$$\mathbf{v_1v_2 + v_1v_3 + v_1v_4 + v_2v_3 + v_2v_4 + v_3v_4 \ = \ \frac{2}{3}s_2} \ , \tag{B15}$$

$$\mathbf{v_1v_2v_3 + v_1v_2v_4 + v_1v_3v_4 + v_2v_3v_4 \ = \ -\frac{c^3}{27}} \ , \tag{B16}$$

$$\mathbf{v_1v_2v_3v_4 \ = \ \frac{c^2s_1 + 3s_2^2}{27}} \ . \tag{B17}$$

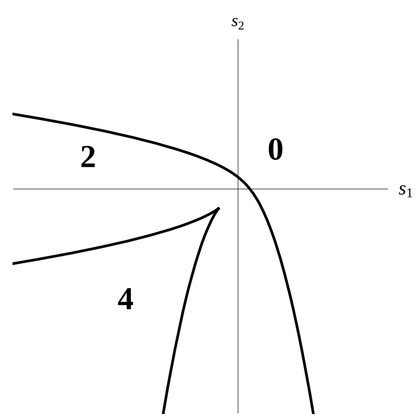

FIG. 6: Caustic curve for the generic hyperbolic umbilic mapping $\mathbf{f}_c$ in equation (B11) or Table I. The number of lensed images for sources in their respective regions is indicated.

We now insert $u_i(v_i)$ into the total signed magnification (B13) and factor the numerator of the resulting expression in powers of s_2:

$$-\,46656\,c\,s_2^3\left(v_1v_2v_3 + v_1v_2v_4 + v_1v_3v_4 + v_2v_3v_4\right) \ , \tag{B18}$$

$$
\begin{aligned}
-\,c\,s_2^2\,\big(2592\,c^3\,(v_1v_2 \;&+\; v_1v_3+v_2v_3+v_1v_4+v_2v_4+v_3v_4)+139968\,\big(v_1^3v_2v_4+v_1v_2^3v_4+v_1^3v_3v_4+v_2^3v_3v_4 \\
&+\; v_1v_3^3v_4+v_2v_3^3v_4+v_1v_2v_4^3+v_1v_3v_4^3+v_2v_3v_4^3+v_1^3v_2v_3+v_1v_2^3v_3+v_1v_2v_3^3\big)\big)\ ,
\end{aligned}
\tag{B19}
$$

$$
\begin{aligned}
c\,s_2\,\big(108\,c^6\,(v_1 \;&+\; v_2+v_3+v_4)+7776\,c^3\,\big(v_1^3v_2+v_1v_2^3+v_1^3v_3+v_2^3v_3+v_1v_3^3+v_2v_3^3+v_1^3v_4+v_2^3v_4+v_3^3v_4 \\
&+\; v_1v_4^3+v_2v_4^3+v_3v_4^3\big)+419904\,\big(v_1^3v_2^3v_3+v_1^3v_2v_3^3+v_1v_2^3v_3^3+v_1^3v_2^3v_4+v_1^3v_3^3v_4 \\
&+\; v_2^3v_3^3v_4+v_1^3v_2v_4^3+v_1v_2^3v_4^3+v_1^3v_3v_4^3+v_2^3v_3v_4^3+v_1v_3^3v_4^3+v_2v_3^3v_4^3\big)\big)\ ,
\end{aligned}
\tag{B20}
$$

$$
\begin{aligned}
-\,4\,c^{10} \;&-\; 324\,c^7\,\big(v_1^3+v_2^3+v_4^3+v_3^3\big)-23328\,c^4\,\big(v_1^3v_3^3+v_2^3v_3^3+v_1^3v_2^3+v_1^3v_4^3+v_2^3v_4^3+v_3^3v_4^3\big) \\
&-\; 1259710\,c\,\big(v_1^3v_2^3v_4^3+v_1^3v_3^3v_4^3+v_2^3v_3^3v_4^3+v_1^3v_2^3v_3^3\big)\ .
\end{aligned}
\tag{B21}
$$

Perusal of these expressions shows that aside from constant factors, the form of each expression in parentheses is once again *identical* to its counterpart in the lensing map case in Appendix A 2. This means that the procedures we employed above, (A52)–(A61), carry through without modification. Of course, as with the elliptic umbilic, we expect different final answers for each term, since the right-hand sides of (B15)–(B17) differ from those of (A45)–(A47). With that said, we once again forgo the labor and only state the final results:

The s_2^3-term: $1728c^4s_2^3$,

The s_2^2-term: $-8640c^4s_2^3$,

The s_2-term: $1728c^6s_1s_2+19008c^4s_2^3$,

The term with no s_2-dependence: $-1728c^6s_1s_2-12096c^4s_2^3$.

We can now verify these terms sum to zero. □

3. Swallowtail

The generic form of a one-parameter family of maps between planes in the neighborhood of a swallowtail critical point can be found in Golubitsky & Guillemin 1973 [20, p. 176]. The resulting map is

$$
\begin{aligned}
s_1 &= uv+cu^2+u^4\ , \\
s_2 &= v\ ,
\end{aligned}
\tag{B22}
$$

and the corresponding magnification of an image (u,v) is

$$
\left(\det(\mathrm{Jac}\ \mathbf{s})\right)^{-1}(u,v) \;=\; \frac{1}{2cu+4u^3+v}\ . \tag{B23}
$$

The caustic curve is shown in Figure 7. The “tail” constitutes the four-image region. We will show that the total signed magnification for all sources inside this region is identically zero.

Let (s_1,s_2) denote an arbitrary source in the four-image region, and (u_i,v_i) the corresponding lensed images. The total signed magnification μ at (s_1,s_2) is

$$
\mu(u_i)=\frac{1}{2cu_1+4u_1^3+s_2}+\frac{1}{2cu_2+4u_2^3+s_2}+\frac{1}{2cu_3+4u_3^3+s_2}+\frac{1}{2cu_4+4u_4^3+s_2}\ , \tag{B24}
$$

where we have used the second lens equation (B22) to substitute $v_i=s_2$ into the magnification of each lensed image. This time the (depressed) quartic is in u,

$$
u^4+cu^2+s_2u-s_1=0\ ,
$$

and the corresponding four equations involving only the u_i are

$$
\mathbf{u_1+u_2+u_3+u_4 \;=\; 0}\ , \tag{B25}
$$

$$
\mathbf{u_1u_2+u_1u_3+u_1u_4+u_2u_3+u_2u_4+u_3u_4 \;=\; c}\ , \tag{B26}
$$

$$
\mathbf{u_1u_2u_3+u_1u_2u_4+u_1u_3u_4+u_2u_3u_4 \;=\; -s_2}\ , \tag{B27}
$$

$$
\mathbf{u_1u_2v_3v_4 \;=\; -s_1}\ . \tag{B28}
$$

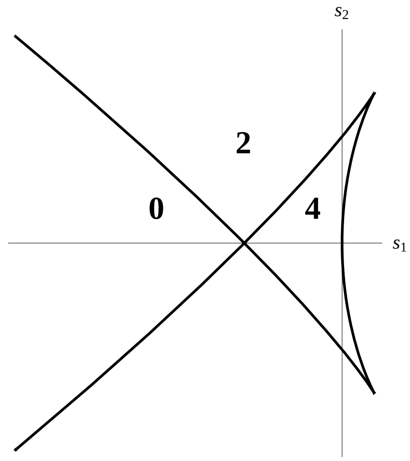

FIG. 7: Caustic curve for the generic swallowtail mapping $\mathbf{f}_c$ in equation (B22) or Table I. The number of lensed images for sources in their respective regions is indicated. Note that in order to produce the tail, we must have $c < 0$ for the given form of the swallowtail in (B22).

In what is now becoming a familiar story, we write (B24) over a common denominator and then factor the resulting numerator in powers of s_2:

$$-4\,s_2^3\ ,$$

$$s_2^2\ \left(6c\,(u_1+u_2+u_3+u_4)+12\left(u_1^3+u_2^3+u_3^3+u_4^3\right)\right)\ ,$$

$$\begin{aligned}s_2\,\Big(8c^2\,&(u_1u_2+u_1u_3+u_2u_3+u_1u_4+u_2u_4+u_3u_4)+16c\left(u_1^3u_2+u_1u_2^3+u_1^3u_3+u_2^3u_3+u_1u_3^3+u_2u_3^3\right.\\ &+\left.u_1^3u_4+u_2^3u_4+u_3^3u_4+u_1u_4^3+u_2u_4^3+u_3u_4^3\right)+32\left(u_1^3u_2^3+u_1^3u_3^3+u_2^3u_3^3+u_1^3u_4^3+u_2^3u_4^3+u_3^3u_4^3\right)\Big)\ ,\end{aligned}$$

$$\begin{aligned}8c^3\,(u_1u_2u_3\ &+\ u_1u_2u_4+u_1u_3u_4+u_2u_3u_4)+16c^2\left(u_1^3u_2u_3+u_1u_2^3u_3+u_1u_2u_3^3+u_1^3u_2u_4+u_1u_2^3u_4\right.\\ &+\ \left.u_1^3u_3u_4+u_2^3u_3u_4+u_1u_3^3u_4+u_2u_3^3u_4+u_1u_2u_4^3+u_1u_3u_4^3+u_2u_3u_4^3\right)+32c\left(u_1^3u_2^3u_3\right.\\ &+\ u_1^3u_2u_3^3+u_1u_2^3u_3^3+u_1^3u_2^3u_4+u_1^3u_3^3u_4+u_2^3u_3^3u_4+u_1^3u_2u_4^3+u_1u_2^3u_4^3+u_1^3u_3u_4^3\\ &+\ \left.u_2^3u_3u_4^3+u_1u_3^3u_4^3+u_2u_3^3u_4^3\right)+64\left(u_1^3u_2^3u_3^3+u_1^3u_2^3u_4^3+u_1^3u_3^3u_4^3+u_2^3u_3^3u_4^3\right)\ .\end{aligned}$$

Perusal of these expressions shows that every polynomial in parentheses has already been calculated in the case of the hyperbolic umbilic. Once again, we expect different final answers for each term, since the right-hand sides of (B26)–(B28) differ from those of (A45)–(A47). With that said, we once again forgo the labor and only state the final results:

The s_2^3-term: $4s_2^3$,

The s_2^2-term: $-36s_2^3$,

The s_2-term: $8c^3s_2+32cs_1s_2+96s_2^3$,

The term with no s_2-dependence: $-8c^3s_2-32cs_1s_2-64s_2^3$.

We can now verify that these terms sum to zero. □

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